

# New associative product of three states generalizing free, monotone, anti-monotone, Boolean, conditionally free and conditionally monotone products

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## Abstract

We define a new independence in three states called indented independence which unifies many independences: free, monotone, anti-monotone, Boolean, conditionally free, conditionally monotone and conditionally anti-monotone independences. This unification preserves the associative laws. Therefore, the central limit theorem, cumulants and moment-cumulant formulae for indented independence also unify those for the above seven independences.

**Keywords:** Conditionally free independence; free independence; monotone independence; Boolean independence; cumulants

**Mathematics Subject Classification:** Primary 46L53, 46L54; secondary 06A07

## 1 Introduction

Several kinds of independences have been discovered since free independence was introduced by D. Voiculescu [29]. After such discoveries, there were attempts to define and classify independence. An exchangeability system introduced by F. Lehner [14] is a very general definition of independence. Other important classes are universal independence and natural independence; the former one was studied by M. Schürmann, R. Speicher and A. Ben Ghorbal and the latter was by N. Muraki [4, 17, 20, 25, 27]. There are also many attempts to interpolate different independences. The conditionally (c- for short hereafter) free independence, initiated by M. Bożejko, M. Leinert and R. Speicher [7, 8], is an important one in that it includes six independences: free [29], Boolean [6, 28], monotone [17, 18], anti-monotone [20], c-monotone [11] and c-anti-monotone independences. While the explicit definition of the last one may not be found in the literature, it can be defined by reversing the order structure of c-monotone independence.

C-free independence and the other six can be formulated as products of states in the free product of algebras with or without identification of units. Important properties of the above mentioned products of states are associative laws. For instance, associativity was crucial in the classification of universal independence, quasi-universal independence and natural independence [4, 17, 20, 25, 27]. The associative laws of the c-free, free, Boolean products are not difficult to prove on the basis of their definitions. However, the associative laws of monotone and c-monotone products (and moreover anti-monotone and c-anti-monotone products) are not trivial. U. Franz

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\*The author is supported by Grant-in-Aid for JSPS Fellows.

proved the associative law of the monotone product in [9] and the author proved the associative law of the c-monotone product in [11]. Associativity is also a central topic of this paper.

We define the c-free product of states [8].

**Definition 1.1.** Let  $I$  be an index set and  $(\mathcal{A}_i, \varphi_i, \psi_i)$  ( $i \in I$ ) be algebraic probability spaces equipped with two states.  $\mathcal{A}_i$  are assumed to be unital. The c-free product of  $(\mathcal{A}_i, \varphi_i, \psi_i)$ , denoted by  $(\mathcal{A}, \varphi, \psi) = *_{i \in I} (\mathcal{A}_i, \varphi_i, \psi_i)$ , is defined by the following rules.  $\mathcal{A} := *_{i \in I} \mathcal{A}_i$  is defined to be the free product with identification of units and  $\psi := *_{i \in I} \psi_i$  the free product of states.  $\varphi$  is defined by the following property.

If  $a_k \in \mathcal{A}_{i_k}$  with  $i_1 \neq \dots \neq i_n$  and  $\psi_{i_k}(a_k) = 0$  for all  $1 \leq k \leq n$ , then

$$\varphi(a_1 \cdots a_n) = \prod_{k=1}^n \varphi_{i_k}(a_k). \quad (1.1)$$

The notation  $i_1 \neq \dots \neq i_n$  means that the neighboring elements are different. We often write only states in such a form as  $(\varphi, \psi) = (\varphi_1, \psi_1) * (\varphi_2, \psi_2)$  and omit algebras when there is no confusion. As is understood by definition, the right component acts on the left. We use the notation  $(\varphi_1 \psi_1 * \varphi_2 \psi_2, \psi_1 * \psi_2) = (\varphi_1, \psi_1) * (\varphi_2, \psi_2)$  to express the action.

We explain the connections to the six independences. Let  $*$ ,  $\diamond$ ,  $\triangleright$  and  $\triangleleft$  be the free, Boolean, monotone and anti-monotone products of states respectively. By definition the free product appears if  $\varphi_i = \psi_i$ ,  $i = 1, 2$ :  $\varphi_1 \varphi_1 * \varphi_2 \varphi_2 = \varphi_1 * \varphi_2$ . To state the connection to Boolean and monotone products, we need to consider the unitization of the algebra. Let  $\mathcal{A}_i^0$  be  $*$ -algebras for  $i = 1, 2$  and  $\mathcal{A}_i$  be their unitizations defined by  $\mathcal{A}_i = \mathbb{C} \oplus \mathcal{A}_i^0$ . Then we can define delta states  $\delta_i$  on  $\mathcal{A}_i$  by  $\delta_i(\lambda + a^0) := \lambda$  for  $\lambda \in \mathbb{C}$  and  $a^0 \in \mathcal{A}_i^0$ . From now on we always assume these unitizations when we use delta states. In this setting, the Boolean product  $\diamond$  appears as  $\varphi_1 \delta_1 * \delta_2 \varphi_2 = \varphi_1 \diamond \varphi_2$ . Moreover, U. Franz proved in [10] that the monotone product (resp. anti-monotone product) appears as  $\varphi_1 \delta_1 * \varphi_2 \varphi_2 = \varphi_1 \triangleright \varphi_2$  (resp.  $\varphi_1 \varphi_1 * \delta_2 \varphi_2 = \varphi_1 \triangleleft \varphi_2$ ). The connection to the monotone product yields nontrivial problems: the associative law of monotone product does not follow from that of the c-free product, nor do the monotone cumulants from the c-free cumulants. The same problem also arises about the anti-monotone product. Motivated by these, the author defined a c-monotone product in [11]. The c-monotone product  $\triangleright$  of pairs of states is defined by

$$(\varphi_1, \psi_1) \triangleright (\varphi_2, \psi_2) := (\varphi_1 \delta_1 * \psi_2 \varphi_2, \psi_1 \triangleright \psi_2).$$

The left component  $\varphi_1 \delta_1 * \psi_2 \varphi_2$  is also denoted by  $\varphi_1 \triangleright_{\psi_2} \varphi_2$ . The c-monotone product is associative, but this is not a consequence of the associativity of the c-free product. The c-monotone product can be seen as a generalization of monotone and Boolean products:  $(\varphi_1, \varphi_1) \triangleright (\varphi_2, \varphi_2) = (\varphi_1 \triangleright \varphi_2, \varphi_1 \triangleright \varphi_2)$  and  $(\varphi_1, \delta_1) \triangleright (\varphi_2, \delta_2) = (\varphi_1 \diamond \varphi_2, \delta_1 * \delta_2)$ . Moreover, the associative laws of monotone and Boolean products are consequences of that of the c-monotone product. A c-anti-monotone product is similarly defined by

$$(\varphi_1, \psi_1) \triangleleft (\varphi_2, \psi_2) := (\varphi_1 \psi_1 * \delta_2 \varphi_2, \psi_1 \triangleleft \psi_2)$$

and the left component  $\varphi_1 \psi_1 * \delta_2 \varphi_2$  is denoted as  $\varphi_1 \triangleleft_{\psi_2} \varphi_2$ .

In this paper we construct an associative product of triples of states which generalizes free, c-free, monotone, anti-monotone, Boolean, c-monotone, c-anti-monotone products. This is defined by

$$(\varphi_1, \psi_1, \theta_1) \curlywedge (\varphi_2, \psi_2, \theta_2) = (\varphi_1 \theta_1 * \psi_2 \varphi_2, \psi_1 \theta_1 * \psi_2 \psi_2, \theta_1 \theta_1 * \psi_2 \theta_2), \quad (1.2)$$

which will be called an *indented product*. In particular, the product

$$(\varphi_1, \psi_1) \curlywedge (\varphi_2, \psi_2) = (\varphi_1 \psi_1 * \varphi_2 \varphi_2, \psi_1 \psi_1 * \varphi_2 \psi_2) \quad (1.3)$$

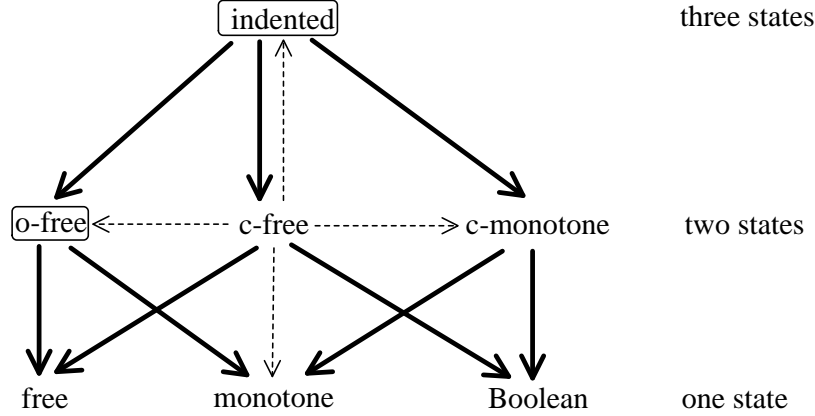


Figure 1: Each arrow means that the initial product generalizes the terminal one. An arrow without dots means that it preserves the associative laws; a dotted arrow means that it does not preserve the associative laws. Anti-monotone and c-anti-monotone products are omitted. Indented and o-free products are new concepts and therefore they are emphasized by rectangles.

is associative. This will be called an *ordered free* (or o-free for simplicity) *product* and denoted by the same symbol  $\lambda$ . While these products are defined by a combinations of c-free products, the associative laws do not follow from that of the c-free product. These situations are summarized in Fig. 1.

Furthermore, o-free and indented products are naturally expected to have connections with the concept of *matricial freeness* introduced by R. Lenczewski recently [15]. We however leave this direction to a future research.

We note that the products

$$(\varphi_1, \psi_1, \theta_1) \prec (\varphi_2, \psi_2, \theta_2) = (\varphi_1 \psi_1 * \theta_2 \varphi_2, \psi_1 \psi_1 * \theta_2 \psi_2, \theta_1 \psi_1 * \theta_2 \theta_2)$$

and

$$(\varphi_1, \psi_1) \prec (\varphi_2, \psi_2) = (\varphi_1 \varphi_1 * \psi_2 \varphi_2, \psi_1 \varphi_1 * \psi_2 \psi_2)$$

are also associative. The structures of these products are equal to the indented product and the ordered free product respectively. We therefore do not mention these two anymore.

We explain the contents of this paper. In Section 2, we characterize the additive and multiplicative convolutions in terms of reciprocal of Cauchy transforms. The reader may wonder how the products (1.2) and (1.3) were found to be associative; therefore we explain the motivation for the definition of (1.3) as an application of the characterizations. Once this is explained, the definition (1.2) will be understood as a natural extension of (1.3). In Section 3 the associative laws of the indented product and o-free product will be proved. In Section 4 we construct a representation of the free product of unital algebras which enables us to calculate the indented and o-free products in terms of operators on a Hilbert space. Motivation for this section comes from the works by D. Avitzour [2], D. Voiculescu [29], M. Bożejko and R. Speicher [8] and M. Popa [21].

The remaining contents are mainly devoted to cumulants. In free probability theory, there have been many researches on combinatorial aspects of cumulants since R. Speicher introduced non-crossing partitions in [26]. In the present paper, a crucial partition structure is “linearly ordered non-crossing partitions” first introduced by N. Muraki [19]. In Section 5, we define cumulants for indented independence. This independence is noncommutative, that is, if  $X$  and  $Y$  are independent,  $Y$  and  $X$  are not necessarily independent; therefore, the corresponding

cumulants should be defined along the line of [13]. Since the associative laws of the seven kinds of products follow from that of the indented product, moment-cumulant formulae for them also follow from indented independence. In particular, we obtain moment-cumulant formula for  $c$ -monotone independence. We then derive differential equations as relations between generating functions of moments and cumulants for single variable. In Section 6 we prove the central limit theorem w.r.t. indented independence. The limit measures are Kesten distributions; this result unifies the limit distributions in the  $c$ -free and  $c$ -monotone cases.

## 2 New convolution of probability measures

We start from the description of additive convolutions of probability measures. This section will be useful for the reader to understand the idea of Section 3.

Let  $\mathbb{C}[z]$  be the unital algebra generated from one indeterminate  $z$  equipped with the operation  $z^* = z$ . Then there is a one-to-one correspondence between a state  $\varphi$  on  $\mathbb{C}[z]$  and a probability measure  $\mu$  defined by  $\int x^n \mu(dx) = \varphi(z^n)$  when the moment sequence  $\{\varphi(z^n)\}_{n=0}^\infty$  is determinate [1]. If a product of algebraic probability spaces  $(\mathcal{A}_1, \varphi_1) \cdot (\mathcal{A}_2, \varphi_2) = (\mathcal{A}_1 * \mathcal{A}_2, \varphi_1 \cdot \varphi_2)$  is given, one can define the associated additive convolution of probability measures. That is, let  $\mathcal{A}_i$  be  $\mathbb{C}[z_i]$  and  $\mu_i$  be the probability measure corresponding to the moments  $\varphi_i(z_i^n)$ . Then the convolution  $\mu_1 \cdot \mu_2$  is defined by the moments  $\varphi_1 \cdot \varphi_2((z_1 + z_2)^n)$ , if the resulting moments are determinate. If the product of states is defined in the category of  $C^*$ -algebras, however, we can only treat probability measures with compact supports and a moment problem is always determinate.

Also we can define a multiplicative convolution for a given product of states. Let  $\mathbb{C}[z, z^{-1}]$  be the unital algebra generated from  $z$  and  $z^{-1}$  satisfying the relation  $z^{-1}z = zz^{-1} = 1$ .  $*$  is defined by extending the definition  $z^* = z^{-1}$  to  $\mathbb{C}[z, z^{-1}]$  so that it becomes anti-linear. We denote by  $\mathcal{P}(\mathbb{T})$  the set of probability measures on  $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ . Then there is a one-to-one correspondence between a state  $\varphi$  on  $\mathbb{C}[z, z^{-1}]$  and a probability measure  $\mu \in \mathcal{P}(\mathbb{T})$  by  $\varphi(z^n) = \int_{\mathbb{T}} \zeta^n \mu(d\zeta)$ ; the reader is referred to Chapter 5 of [1]. Let  $(\mathbb{C}[z_i, z_i^{-1}], \varphi_i)$  be a (algebraic) probability space and  $\mu_i \in \mathcal{P}(\mathbb{T})$  be the probability measure corresponding to the moments  $\varphi_i(z_i^n)$ . Then the convolution  $\mu_1 \cdot \mu_2$  is defined by the moments  $\varphi_1 \cdot \varphi_2((z_1 z_2)^n)$ .

When we consider a product of algebraic probability spaces with two states, we can define a convolution  $(\mu, \nu) = (\mu_1, \nu_1) \cdot (\mu_2, \nu_2)$  similarly. Three states or more can also be treated similarly.

The Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z - x}, \quad z \in \mathbb{C} \setminus \mathbb{R} \quad (2.1)$$

of a probability measure  $\mu$  is useful in characterizing convolutions of probability measures. In addition, its reciprocal

$$F_\mu(z) = \frac{1}{G_\mu(z)} \quad (2.2)$$

is also important.

Now we review the complex analytic characterization of the  $c$ -free convolution [7]. For simplicity, we only consider probability measures with compact supports. We define the  $R$ -transform [30] and the  $c$ -free  $R$ -transform [7] by

$$\frac{1}{G_\nu(z)} = z - R_\nu(G_\nu(z)), \quad (2.3)$$

$$\frac{1}{G_\mu(z)} = z - R_{(\mu, \nu)}(G_\nu(z)). \quad (2.4)$$

The coefficients  $R_n(\mu, \nu)$  in  $R_{(\mu, \nu)}(z) = \sum_{n=1}^{\infty} R_n(\mu, \nu) z^{n-1}$  are called the c-free cumulants and the coefficients  $R_n(\nu)$  in  $R_\nu(z) = \sum_{n=1}^{\infty} R_n(\nu) z^{n-1}$  are called the free cumulants. The above relations can be formulated in terms of  $F_\mu(z)$ ,  $\phi_{(\mu, \nu)}(z) := R_{(\mu, \nu)}(\frac{1}{z})$  and  $\phi_\nu(z) := R_\nu(\frac{1}{z})$ :

$$F_\nu(z) = z - \phi_\nu(F_\nu(z)), \quad (2.5)$$

$$F_\mu(z) = z - \phi_{(\mu, \nu)}(F_\nu(z)). \quad (2.6)$$

The c-free convolution of  $(\mu_1, \nu_1)$  and  $(\mu_2, \nu_2)$ , denoted as  $(\mu, \nu) = (\mu_1, \nu_1) \boxplus (\mu_2, \nu_2)$ , is characterized by

$$\phi_\nu(z) = \phi_{\nu_1}(z) + \phi_{\nu_2}(z), \quad (2.7)$$

$$\phi_{(\mu, \nu)}(z) = \phi_{(\mu_1, \nu_1)}(z) + \phi_{(\mu_2, \nu_2)}(z). \quad (2.8)$$

(2.7) can be written as follows.

$$F_{\nu_1 \boxplus \nu_2}^{-1}(z) = F_{\nu_1}^{-1}(z) + F_{\nu_2}^{-1}(z) - z. \quad (2.9)$$

In view of the notation of the c-free product of states, it is natural to denote by  $(\mu_1 \nu_1 \boxplus \nu_2 \mu_2, \nu_1 \boxplus \nu_2)$  the c-free convolution of  $(\mu_1, \nu_1)$  and  $(\mu_2, \nu_2)$ .

Now we explain the motivation for this paper. If we try to find a nontrivial associative convolution and independence, it is natural to investigate  $\mu_1 \nu_1 \boxplus \mu_2$  instead of the c-monotone convolution  $\mu_1 \delta_0 \boxplus \mu_2$ . First we characterize the convolution.

The following result was proved in [3] including measures with unbounded supports. We state the result only for compactly supported measures in the sense of formal power series.

**Proposition 2.1.** *For compactly supported probability measures  $\mu_i, \nu_i, i = 1, 2$ , the convolution  $\mu_1 \nu_1 \boxplus \mu_2$  is characterized by*

$$F_{\mu_1 \nu_1 \boxplus \mu_2} = F_{\mu_1} \circ F_{\nu_1}^{-1} \circ F_{\nu_1 \boxplus \mu_2} + F_{\mu_2} \circ F_{\nu_2}^{-1} \circ F_{\nu_1 \boxplus \mu_2} - F_{\nu_1 \boxplus \mu_2}$$

in the sense of formal power series.

**Corollary 2.2.** *For compactly supported probability measures  $\mu_i, \nu_i, i = 1, 2$ , the convolution  $\mu_1 \nu_1 \boxplus \mu_2$  is characterized by*

$$F_{\mu_1 \nu_1 \boxplus \mu_2} = F_{\mu_1} \circ F_{\nu_1}^{-1} \circ F_{\nu_1 \boxplus \mu_2}.$$

in the sense of formal power series.

We look for an associative convolution of pairs of probability measures of the form  $(\mu_1 \nu_1 \boxplus \mu_2, \lambda)$ , where  $\lambda = \lambda(\mu_1, \mu_2, \nu_1, \nu_2)$  is a probability measure on  $\mathbb{R}$  depending on  $\mu_1, \mu_2, \nu_1, \nu_2$ . It turns out that  $\lambda$  should be taken to be  $\nu_1 \nu_1 \boxplus \mu_2 \nu_2$ . We explain how to prove this. We assume that a product  $\square$  defined by  $(\mu_1, \nu_1) \square (\mu_2, \nu_2) = (\mu_1 \nu_1 \boxplus \mu_2, \lambda)$  is associative. Then the associativity implies that

$$(\mu_1 \nu_1 \boxplus \mu_2) \lambda \boxplus \mu_3 \mu_3 = \mu_1 \nu_1 \boxplus \mu_2 \nu_2 \boxplus \mu_3 \mu_3 (\mu_2 \nu_2 \boxplus \mu_3 \mu_3).$$

By the way, Proposition 2.1 implies that

$$\begin{aligned} F_{(\mu_1 \nu_1 \boxplus \mu_2) \lambda \boxplus \mu_3 \mu_3} &= F_{\mu_1 \nu_1 \boxplus \mu_2} \circ F_\lambda^{-1} \circ F_{\lambda \boxplus \mu_3} \\ &= F_{\mu_1} \circ F_{\nu_1}^{-1} \circ F_{\nu_1 \boxplus \mu_2} \circ F_\lambda^{-1} \circ F_{\lambda \boxplus \mu_3} \end{aligned}$$

and

$$F_{\mu_1 \nu_1 \boxplus (\mu_2 \nu_2 \boxplus \mu_3 \mu_3)} (\mu_2 \nu_2 \boxplus \mu_3 \mu_3) = F_{\mu_1} \circ F_{\nu_1}^{-1} \circ F_{\nu_1 \boxplus (\mu_2 \nu_2 \boxplus \mu_3 \mu_3)}.$$

Therefore, it holds that  $F_{\nu_1 \boxplus \mu_2} \circ F_{\lambda}^{-1} \circ F_{\lambda \boxplus \mu_3} = F_{\nu_1 \boxplus (\mu_2 \nu_2 \boxplus \mu_3 \mu_3)}$ , or equivalently,

$$F_{\lambda \boxplus \mu_3}^{-1} \circ F_{\lambda} \circ F_{\nu_1 \boxplus \mu_2}^{-1} = F_{\nu_1 \boxplus (\mu_2 \nu_2 \boxplus \mu_3 \mu_3)}^{-1}. \quad (2.10)$$

The left hand side is

$$\begin{aligned} F_{\lambda \boxplus \mu_3}^{-1} \circ F_{\lambda} \circ F_{\nu_1 \boxplus \mu_2}^{-1} &= (z + F_{\mu_3}^{-1} \circ F_{\lambda} - F_{\lambda}) \circ F_{\nu_1 \boxplus \mu_2}^{-1} \\ &= F_{\nu_1 \boxplus \mu_2}^{-1} + F_{\mu_3}^{-1} \circ F_{\lambda} \circ F_{\nu_1 \boxplus \mu_2}^{-1} - F_{\lambda} \circ F_{\nu_1 \boxplus \mu_2}^{-1} \end{aligned} \quad (2.11)$$

by using (2.9). On the other hand we have

$$\begin{aligned} F_{\nu_1 \boxplus (\mu_2 \nu_2 \boxplus \mu_3 \mu_3)}^{-1} &= F_{\nu_1}^{-1} + F_{(\mu_2 \nu_2 \boxplus \mu_3 \mu_3)}^{-1} - z \\ &= F_{\nu_1}^{-1} + F_{\nu_2 \boxplus \mu_3}^{-1} \circ F_{\nu_2} \circ F_{\mu_2}^{-1} - z \\ &= F_{\nu_1}^{-1} + F_{\mu_2}^{-1} + F_{\mu_3}^{-1} \circ F_{\nu_2} \circ F_{\mu_2}^{-1} - F_{\nu_2} \circ F_{\mu_2}^{-1} - z \\ &= F_{\nu_1 \boxplus \mu_2}^{-1} + F_{\mu_3}^{-1} \circ F_{\nu_2} \circ F_{\mu_2}^{-1} - F_{\nu_2} \circ F_{\mu_2}^{-1}. \end{aligned}$$

Then (2.10) implies that  $F_{\mu_3}^{-1} \circ F_{\nu_2} \circ F_{\mu_2}^{-1} - F_{\nu_2} \circ F_{\mu_2}^{-1} = F_{\mu_3}^{-1} \circ F_{\lambda} \circ F_{\nu_1 \boxplus \mu_2}^{-1} - F_{\lambda} \circ F_{\nu_1 \boxplus \mu_2}^{-1}$ . This is satisfied if we define

$$F_{\lambda} = F_{\nu_1 \nu_1 \boxplus \mu_2 \nu_2}.$$

Thus we can determine  $\lambda$ . The above discussion implies that

$$(\mu_1 \nu_1 \boxplus \mu_2 \nu_2)_{(\nu_1 \nu_1 \boxplus \mu_2 \nu_2) \boxplus \mu_3 \mu_3} = \mu_1 \nu_1 \boxplus_{(\mu_2 \nu_2 \boxplus \mu_3 \mu_3)} (\mu_2 \nu_2 \boxplus \mu_3 \mu_3).$$

If we replace  $\mu_1, \mu_2, \mu_3, \nu_1$  and  $\nu_2$  respectively with  $\nu_3, \nu_2, \nu_1, \mu_3$  and  $\mu_2$ , then we have

$$(\nu_1 \nu_1 \boxplus \mu_2 \nu_2)_{(\nu_1 \nu_1 \boxplus \mu_2 \nu_2) \boxplus \mu_3 \nu_3} = \nu_1 \nu_1 \boxplus_{(\mu_2 \nu_2 \boxplus \mu_3 \mu_3)} (\nu_2 \nu_2 \boxplus \mu_3 \nu_3).$$

These two relations imply the associative law of the convolution. We can prove the following results.

**Proposition 2.3.** *Let  $\mu_i, \nu_i$  ( $i = 1, 2$ ) be compactly supported probability measures.*

(1) *The convolution  $\lambda$  defined by*

$$(\mu_1, \nu_1) \lambda (\mu_2, \nu_2) = (\mu_1 \nu_1 \boxplus \mu_2 \nu_2, \nu_1 \nu_1 \boxplus \mu_2 \nu_2)$$

*is associative.*

(2) *The convolution  $\lambda$  defined by*

$$(\lambda_1, \mu_1, \nu_1) \lambda (\lambda_2, \mu_2, \nu_2) = (\lambda_1 \nu_1 \boxplus \mu_2 \lambda_2, \mu_1 \nu_1 \boxplus \mu_2 \nu_2, \nu_1 \nu_1 \boxplus \mu_2 \nu_2)$$

*is associative.*

(1) was proved in the above. The proof of (2) is similar to that of (1). We will however prove these results more generally in the next section.

**Definition 2.4.** (1) The convolution defined in Proposition 2.3 (1) is called an additive ordered free (o-free) convolution.

(2) The convolution defined in Proposition 2.3 (2) is called an additive indented convolution.

These convolutions are noncommutative. The latter convolution generalizes many convolutions; this will be explained after Definition 3.4.

Next we consider the multiplicative convolution  $\mu_{1\nu_1} \boxtimes_{\mu_2} \mu_2$ . The Cauchy transform is now defined by

$$G_\mu(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\mathbb{T}} \zeta^n \nu(d\zeta) = \int_{\mathbb{T}} \frac{\mu(d\zeta)}{z - \zeta} \text{ for } |z| > 1.$$

As was used effectively in [5, 12],  $\eta_\mu(z) = 1 - \frac{z}{G_\mu(\frac{1}{z})}$ ,  $|z| < 1$  is important also in this paper. We define  $\tilde{R}_{(\mu,\nu)}(z) := zR_{(\mu,\nu)}(z)$  and  $\tilde{R}_\mu(z) := zR_\mu(z)$  which were used in [22] without tildes. The relations (2.3) and (2.4) become

$$\tilde{R}_\nu\left(\frac{z}{1 - \eta_\nu(z)}\right) = \frac{\eta_\nu(z)}{1 - \eta_\nu(z)}, \quad (2.12)$$

$$\tilde{R}_{(\mu,\nu)}\left(\frac{z}{1 - \eta_\nu(z)}\right) = \frac{\eta_\mu(z)}{1 - \eta_\nu(z)}. \quad (2.13)$$

The multiplicative c-free convolution of probability measures on  $\mathbb{T}$  has been characterized in [22] as follows. Let  $T_{(\mu,\nu)}$  be defined by  $T_{(\mu,\nu)}(z) = \frac{\tilde{R}_{(\mu,\nu)}(\tilde{R}_\nu^{-1}(z))}{\tilde{R}_\nu^{-1}(z)}$  and  $T_\nu$  by  $T_\nu(z) = T_{(\nu,\nu)}(z) = \frac{z}{\tilde{R}_\nu^{-1}(z)}$  for  $\mu, \nu \in \mathcal{P}(\mathbb{T})$  such that  $m_1(\nu) = \int_{\mathbb{T}} \zeta \nu(d\zeta) \neq 0$ . The multiplicative c-free convolution  $(\mu_{1\nu_1} \boxtimes_{\nu_2} \mu_2, \nu_1 \boxtimes \nu_2) = (\mu_1, \nu_1) \boxtimes (\mu_2, \nu_2)$  of  $(\mu_1, \nu_1)$  and  $(\mu_2, \nu_2)$  ( $\mu_i, \nu_i \in \mathcal{P}(\mathbb{T})$ ,  $m_1(\nu_i) \neq 0$ ) is characterized by

$$T_{(\mu_{1\nu_1} \boxtimes_{\nu_2} \mu_2, \nu_1 \boxtimes \nu_2)}(z) = T_{(\mu_1, \nu_1)}(z) T_{(\mu_2, \nu_2)}(z), \quad (2.14)$$

$$T_{\nu_1 \boxtimes \nu_2}(z) = T_{\nu_1}(z) T_{\nu_2}(z). \quad (2.15)$$

The multiplicative c-free convolution can be characterized in terms of the transform  $\eta_\mu$ ; this enables us to prove the associative laws of multiplicative convolutions coming from (1.2) and (1.3).

**Proposition 2.5.** *The left component of the convolution  $(\mu_1, \nu_1) \cdot (\mu_2, \nu_2) = (\mu_{1\nu_1} \boxtimes_{\nu_2} \mu_2, \nu_1 \boxtimes \nu_2)$  is characterized by*

$$\eta_{\mu_{1\nu_1} \boxtimes_{\nu_2} \mu_2}(z) = \frac{\eta_{\mu_1} \circ \eta_{\nu_1}^{-1} \circ \eta_{\nu_1 \boxtimes \nu_2}(z) \eta_{\mu_2} \circ \eta_{\nu_2}^{-1} \circ \eta_{\nu_1 \boxtimes \nu_2}(z)}{\eta_{\nu_1 \boxtimes \nu_2}(z)} \quad (2.16)$$

in a neighborhood of 0 for  $\mu_i, \nu_i \in \mathcal{P}(\mathbb{T})$ ,  $m_1(\nu_i) \neq 0$ ,  $i = 1, 2$ .

**Corollary 2.6.** *The equality*

$$\eta_{\mu_{1\nu_1} \boxtimes_{\nu_2} \mu_2} = \eta_{\mu_1} \circ \eta_{\nu_1}^{-1} \circ \eta_{\nu_1 \boxtimes \nu_2} \quad (2.17)$$

holds in a neighborhood of 0 for  $\mu_i, \nu_i \in \mathcal{P}(\mathbb{T})$ ,  $m_1(\nu_i) \neq 0$ .

*Proof.*  $\tilde{R}_{\nu_i}$ ,  $\eta_{\nu_i}$  ( $i = 1, 2$ ),  $\tilde{R}_{\nu_1 \boxtimes \nu_2}$  and  $\eta_{\nu_1 \boxtimes \nu_2}$  are all invertible in a neighborhood of 0 since  $m_1(\nu_i) \neq 0$ . From (2.14) and (2.15) it follows that

$$z\tilde{R}_{(\mu_{1\nu_1} \boxtimes_{\nu_2} \mu_2, \nu_1 \boxtimes \nu_2)}(\tilde{R}_{\nu_1 \boxtimes \nu_2}^{-1}(z)) = \tilde{R}_{(\mu_1, \nu_1)}(\tilde{R}_{\nu_1}^{-1}(z)) \tilde{R}_{(\mu_2, \nu_2)}(\tilde{R}_{\nu_2}^{-1}(z)). \quad (2.18)$$

We define new variables  $u$ ,  $v$  and  $w$  by

$$\tilde{R}_{\nu_1}^{-1}(z) = \frac{u}{1 - \eta_{\nu_1}(u)}, \quad (2.19)$$

$$\tilde{R}_{\nu_2}^{-1}(z) = \frac{v}{1 - \eta_{\nu_2}(v)}, \quad (2.20)$$

$$\tilde{R}_{\nu_1 \boxtimes \nu_2}^{-1}(z) = \frac{w}{1 - \eta_{\nu_1 \boxtimes \nu_2}(w)}. \quad (2.21)$$

These equalities, combined with (2.12) and (2.13), become

$$z = \frac{\eta_{\nu_1}(u)}{1 - \eta_{\nu_1}(u)} = \frac{\eta_{\nu_2}(v)}{1 - \eta_{\nu_2}(v)} = \frac{\eta_{\nu_1 \boxtimes \nu_2}(w)}{1 - \eta_{\nu_1 \boxtimes \nu_2}(w)}, \quad (2.22)$$

and therefore we obtain  $\eta_{\nu_1 \boxtimes \nu_2}(w) = \eta_{\nu_1}(u) = \eta_{\nu_2}(v)$ . (2.18) then implies that

$$z \frac{\eta_{\mu_1 \nu_1 \boxtimes \nu_2 \mu_2}(w)}{1 - \eta_{\nu_1 \boxtimes \nu_2}(w)} = \frac{\eta_{\mu_1}(u)}{1 - \eta_{\nu_1}(u)} \frac{\eta_{\mu_2}(v)}{1 - \eta_{\nu_2}(v)}.$$

Since

$$\frac{z}{1 - \eta_{\nu_1 \boxtimes \nu_2}(w)} = \frac{z^2}{\eta_{\nu_1 \boxtimes \nu_2}(w)} = \frac{\eta_{\nu_1 \boxtimes \nu_2}(w)}{(1 - \eta_{\nu_1}(u))(1 - \eta_{\nu_2}(v))},$$

the claim follows.

Next we prove the corollary. This is the case if  $m_1(\nu_i) \neq 0$  for  $i = 1, 2$ . Now we only assume that  $m_1(\nu_1) \neq 0$ . We can find a sequence  $\mu^{(n)}$  with  $m_1(\mu^{(n)}) \neq 0$  which converges weakly to  $\mu_2$ . We note that the weak convergence is equivalent to the convergence of the moments, and also equivalent to the pointwise convergence of the Cauchy transforms.  $\square$

It is worthy to note the similarity between Proposition 2.1 and Proposition 2.5. If  $m_1(\mu) \neq 0$ , we define  $f_\mu = \log \circ \eta_\mu \circ \exp$  and then Proposition 2.5 becomes

$$f_{\mu_1 \nu_1 \boxtimes \nu_2 \mu_2} = f_{\mu_1} \circ f_{\nu_1}^{-1} \circ f_{\nu_1 \boxtimes \nu_2} + f_{\mu_2} \circ f_{\nu_2}^{-1} \circ f_{\nu_1 \boxtimes \nu_2} - f_{\nu_1 \boxtimes \nu_2},$$

which is the same form as Proposition 2.1. Therefore we can prove the associative laws of the multiplicative convolutions defined in the same way as the additive convolutions. We however do not mention the multiplicative convolutions anymore in this paper.

### 3 Indented independence and ordered free independence

In view of the previous section, it is expected that the product of states  $(\varphi_1, \psi_1) \succ (\varphi_2, \psi_2) := (\varphi_1 \psi_1 * \varphi_2, \psi_1 \psi_1 * \varphi_2 \psi_2)$  is also associative. This is the case as we shall see. More generally, the product  $(\varphi_1, \psi_1, \theta_1) \succ (\varphi_2, \psi_2, \theta_2) = (\varphi_{1\theta_1} * \varphi_2, \psi_{1\theta_1} * \psi_2, \theta_{1\theta_1} * \theta_2)$  is also associative. To prove these, we need some computation rules of mixed moments of c-free products.

We note that by definition  $\varphi_{1\psi_1} * \varphi_2(a_1 \cdots a_n) = 0$  holds whenever  $a_k \in \mathcal{A}_{i_k}$ ,  $i_1 \neq \cdots \neq i_n$ ,  $n \geq 2$  for  $i_k = 1, 2$ ,  $\psi_1(a_k) = 0$  for all  $1 \leq k \leq n$  such that  $i_k = 1$  and  $\varphi_2(a_k) = 0$  for all  $1 \leq k \leq n$  such that  $i_k = 2$ . More strongly, the following properties hold.

**Lemma 3.1.** *Let  $\varphi$  be a state on  $\mathcal{A}_1 * \mathcal{A}_2$  with marginal distributions  $\varphi_1$  on  $\mathcal{A}_1$  and  $\varphi_2$  on  $\mathcal{A}_2$ . Then  $\varphi = \varphi_{1\psi_1} * \varphi_2$  if and only if the following properties hold.*

- (1) *If  $a \in \mathcal{A}_1$  and  $b \in \mathcal{A}_2$  then  $\varphi(ab) = \varphi_1(a)\varphi_2(b) = \varphi(ba)$ .*
- (2) *If  $a_1, a_2 \in \mathcal{A}_1$  and  $b_1, b_2 \in \mathcal{A}_2$  then  $\varphi(a_1 b_1 a_2) = \varphi_2(b_1)\varphi_1(a_1 a_2)$  and  $\varphi(b_1 a_1 b_2) = \psi_1(a_1)(\varphi_2(b_1 b_2) - \varphi_2(b_1)\varphi_2(b_2)) + \varphi_2(b_1)\varphi_1(a_1)\varphi_2(b_2)$ .*
- (3) *For  $n \geq 4$ ,  $\varphi(a_1 \cdots a_n) = 0$  whenever  $a_k \in \mathcal{A}_{i_k}$ ,  $i_1 \neq \cdots \neq i_n$ ,  $i_k = 1, 2$ ,  $\psi_1(a_k) = 0$  for all  $2 \leq k \leq n - 1$  such that  $i_k = 1$  and  $\varphi_2(a_k) = 0$  for all  $2 \leq k \leq n - 1$  such that  $i_k = 2$ . In other words, the conditions on  $a_1$  and  $a_n$  are not needed.*

*Proof.* If  $\varphi$  satisfies the properties above, it is immediate that  $\varphi = \varphi_{1\psi_1} * \varphi_2$ . We assume that  $\varphi = \varphi_{1\psi_1} * \varphi_2$ . We only prove (3) since (1) and (2) follow by simple computation. For simplicity,



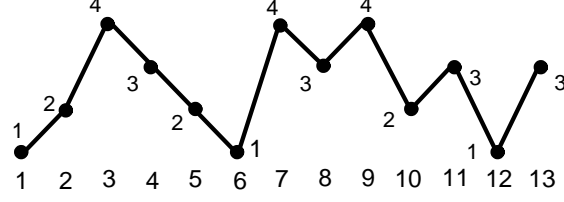


Figure 2:  $i_1 = i_6 = i_{12} = 1$ ,  $i_2 = i_5 = i_{10} = 2$ ,  $i_4 = i_8 = i_{11} = i_{13} = 3$ ,  $i_3 = i_7 = i_9 = 4$ .

we define  $\psi = \psi_1 \psi_1 * \psi_2 \psi_2$ . Let  $\lambda_k$  be  $\varphi(a_k)$  if  $a_k \in \mathcal{A}_1$  and be  $\psi(a_k)$  if  $a_k \in \mathcal{A}_2$ . Then

$$\begin{aligned} \varphi(a_1 \cdots a_n) &= \varphi((a_1 - \lambda_1 + \lambda_1)a_2 \cdots a_{n-1}(a_n - \lambda_n + \lambda_n)) \\ &= \varphi((a_1 - \lambda_1)a_2 \cdots a_{n-1}(a_n - \lambda_n)) + \lambda_1 \varphi(a_2 \cdots a_{n-1}(a_n - \lambda_n)) \\ &\quad + \lambda_n \varphi((a_1 - \lambda_1)a_2 \cdots a_{n-1}) + \lambda_1 \lambda_n \varphi(a_2 \cdots a_{n-1}) \\ &= 0 \end{aligned}$$

under the assumptions on  $a_k$ . □

We now consider general c-free products. In this case we need to put a condition on  $a_1$  or  $a_n$ .

**Lemma 3.2.** *A state  $\varphi$  on  $\mathcal{A}_1 * \mathcal{A}_2$ , having the marginal distributions  $\varphi_1$  on  $\mathcal{A}_1$  and  $\varphi_2$  on  $\mathcal{A}_2$ , is equal to  $\varphi_1 \psi_1 * \psi_2 \varphi_2$  if and only if the following properties hold.*

- (1) *If  $a \in \mathcal{A}_1$  and  $b \in \mathcal{A}_2$  then  $\varphi(ab) = \varphi_1(a)\varphi_2(b) = \varphi(ba)$ .*
- (2) *For  $n \geq 3$ ,  $\varphi(a_1 \cdots a_n) = 0$  holds if  $a_k \in \mathcal{A}_{i_k}$ ,  $i_1 \neq \cdots \neq i_n$ ,  $\varphi_{i_1}(a_1) = 0$  and  $\psi_{i_k}(a_k) = 0$  for all  $2 \leq k \leq n-1$ .*

Moreover, (2) can be replaced by an alternative condition where the kernel of the left edge is replaced by the kernel of the right edge:

- (2')  *$\varphi(a_1 \cdots a_n) = 0$  holds if  $a_k \in \mathcal{A}_{i_k}$ ,  $i_1 \neq \cdots \neq i_n$ ,  $\varphi_{i_n}(a_n) = 0$  and  $\psi_{i_k}(a_k) = 0$  for all  $2 \leq k \leq n-1$ .*

*Proof.* We do not consider the condition (2') since the difference from the condition (2) is only the replacement of  $a_1$  by  $a_n$ .

If  $\varphi$  satisfies the conditions (1) and (2), by definition  $\varphi = \varphi_1 \psi_1 * \psi_2 \varphi_2$ . Conversely, we assume that  $\varphi = \varphi_1 \psi_1 * \psi_2 \varphi_2$ . We denote by  $\psi$  the state  $\psi_1 * \psi_2$ . This proof is similar to that of Lemma 3.1. (1) follows easily. Under the conditions on  $a_k$  in (2), we have

$$\begin{aligned} \varphi(a_1 \cdots a_n) &= \varphi((a_1 - \psi(a_1))a_2 \cdots a_{n-1}(a_n - \psi(a_n))) + \psi(a_1)\psi(a_n)\varphi(a_2 \cdots a_{n-1}) \\ &\quad + \psi(a_1)\varphi(a_2 \cdots a_{n-1}(a_n - \psi(a_n))) + \psi(a_n)\varphi((a_1 - \psi(a_1))a_2 \cdots a_{n-1}) \\ &= (\varphi(a_1) - \psi(a_1))\varphi(a_2) \cdots \varphi(a_{n-1})(\varphi(a_n) - \psi(a_n)) + \psi(a_1)\psi(a_n)\varphi(a_2) \cdots \varphi(a_{n-1}) \\ &\quad + \psi(a_1)\varphi(a_2) \cdots \varphi(a_{n-1})(\varphi(a_n) - \psi(a_n)) + \psi(a_n)(\varphi(a_1) - \psi(a_1))\varphi(a_2) \cdots \varphi(a_{n-1}) \\ &= 0, \end{aligned}$$

since  $\varphi(a_1) = 0$ . □

Before proving the main theorem, we prepare notation. We identify  $(\mathcal{A}_1 * \mathcal{A}_2) * \mathcal{A}_3$  with  $\mathcal{A}_1 * (\mathcal{A}_2 * \mathcal{A}_3)$  by the natural isomorphism and denote it by  $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3$ . Similarly we define  $\mathcal{A}_1 * \cdots * \mathcal{A}_n$  for any  $n \geq 3$ , including  $n = \infty$ . Let  $\mathcal{A} := *_{k \geq 1} \mathcal{A}_k$  be the free product of unital algebras with identification of units. We say that  $x \in \mathcal{A}$  is a *word* (of length  $n$ ) if  $x$  is of the form  $x = a_1 \cdots a_n$ , where  $a_k \in \mathcal{A}_{i_k}$ ,  $i_1 \neq \cdots \neq i_n$ . We visualize a word as in Fig. 2. More

precisely, let  $D := \{(i_1, \dots, i_n); n \in \mathbb{N}, i_k \in \mathbb{N}, 1 \leq k \leq n, i_1 \neq \dots \neq i_n\}$ . Then we can see  $(i_1, \dots, i_n) \in D$  as the function  $i_k$  of  $k$  and the Fig. 2 is its graph. For each  $(i_1, \dots, i_n) \in D$  with  $n \geq 2$ ,  $k$  ( $2 \leq k \leq n-1$ ) is called a *peak* (resp. a *bottom*) if  $i_{k-1} < i_k > i_{k+1}$  (resp.  $i_{k-1} > i_k < i_{k+1}$ ). For  $k=1$  and  $n$ , we also define a peak and a bottom in the natural sense. Let  $P(i_1, \dots, i_n)$  be the set of all peaks and  $B(i_1, \dots, i_n)$  be the set of all bottoms. For instance,  $P(i_1, \dots, i_{13}) = \{3, 7, 9, 11, 13\}$  and  $B(i_1, \dots, i_{13}) = \{1, 6, 8, 10, 12\}$  in Fig. 2.

**Theorem 3.3.** (1) The product  $\lambda$  defined by  $(\varphi_1, \psi_1) \lambda (\varphi_2, \psi_2) := (\varphi_1 \psi_1 *_{\varphi_2} \varphi_2, \psi_1 \psi_1 *_{\varphi_2} \psi_2)$  is associative.

(2) The product  $\lambda$  defined by  $(\varphi_1, \psi_1, \theta_1) \lambda (\varphi_2, \psi_2, \theta_2) = (\varphi_1 \theta_1 *_{\psi_2} \varphi_2, \psi_1 \theta_1 *_{\psi_2} \psi_2, \theta_1 \theta_1 *_{\psi_2} \theta_2)$  is associative.

*Proof.* (1) Let  $\mathcal{A}_i$  be unital algebras,  $(i_1, \dots, i_n) \in D$  and  $a_1 \dots a_n$  a word of  $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3$  such that  $a_k \in \mathcal{A}_{i_k}$  for all  $k$ . We note that  $1 \leq i_k \leq 3$  now. What we need to prove is that

$$(\varphi_1 \psi_1 *_{\varphi_2} \varphi_2)_{(\psi_1 \psi_1 *_{\varphi_2} \psi_2)} *_{\varphi_3} \varphi_3(a_1 \dots a_n) = \varphi_1 \psi_1 *_{(\varphi_2 \psi_2 *_{\varphi_3} \varphi_3)} (\varphi_2 \psi_2 *_{\varphi_3} \varphi_3)(a_1 \dots a_n). \quad (3.1)$$

We assume that  $n \geq 2$ . We can moreover assume that  $a_k \in \text{Ker } \varphi_{i_k}$  for  $k \in P(i_1, \dots, i_n)$  and that  $a_k \in \text{Ker } \psi_{i_k}$  for  $k \in B(i_1, \dots, i_n)$ ; otherwise  $a_1 \dots a_n$  can be decomposed into the sum of such words. To calculate the quantity  $(\varphi_1 \psi_1 *_{\varphi_2} \varphi_2)_{(\psi_1 \psi_1 *_{\varphi_2} \psi_2)} *_{\varphi_3} \varphi_3(a_1 \dots a_n)$ , the numbers 1 and 2 appearing continuously in the sequence  $(i_1, \dots, i_n)$  should be unified. We denote this by parentheses: for instance, the sequence (13232121313212) is reduced to ((1)3(2)3(2121)3(1)3(212)). We omitted commas for simplicity. We write the reduced sequence as  $(I_1 3 I_2 3 \dots)$  or  $(3 I_1 3 I_2 \dots)$ , where  $I_k$  is a sequence of 1 and 2 with different neighboring numbers. We denote  $I_k$  by  $I_k = (i_{\alpha(k)}, \dots, i_{\omega(k)})$ ,  $\alpha(k) \leq \omega(k)$ . Apparently  $\varphi_3(a_k) = 0$  if  $i_k = 3$ , since such a  $k$  is a peak. We can prove that  $\psi_1 \psi_1 *_{\varphi_2} \psi_2 (\prod_{k=\alpha(r)}^{\omega(r)} a_k) = 0$  for each  $r$  by using Lemma 3.1. More precisely, we divide the situation into some cases. (a) If the length  $\omega(r) - \alpha(r) + 1$  is larger than three, then  $\alpha(r) + 1, \dots, \omega(r) - 1$  are all peaks or bottoms. Therefore,  $\psi_1 \psi_1 *_{\varphi_2} \psi_2 (\prod_{k=\alpha(r)}^{\omega(r)} a_k) = 0$  by Lemma 3.1 (3). (b) If the length is three,  $I_r = (i_{\alpha(r)}, i_{\alpha(r)+1}, i_{\alpha(r)+2})$  is either (121) or (212). If  $I_r = (121)$ , all the three points  $\alpha(r), \alpha(r)+1, \alpha(r)+2$  are peaks or bottoms since  $(i_1 \dots i_n)$  is of the form  $(\dots 3 I_r 3 \dots)$ , and therefore  $\psi_1 \psi_1 *_{\varphi_2} \psi_2 (\prod_{k=\alpha(r)}^{\omega(r)} a_k) = 0$  by Lemma 3.1 (2). If  $I_r = (212)$ , then the mid point  $\alpha(r) + 1$  is a bottom. Again by Lemma 3.1 (2)  $\psi_1 \psi_1 *_{\varphi_2} \psi_2 (\prod_{k=\alpha(r)}^{\omega(r)} a_k) = \psi_1(a_{\alpha(r)+1}) \psi_2(a_{\alpha(r)} a_{\alpha(r)+2}) = 0$ . (c) If the length is two, then  $I_r$  is either (12) or (21). In both cases one of the points  $\alpha(r), \alpha(r) + 1$  is a bottom, and hence,  $\psi_1 \psi_1 *_{\varphi_2} \psi_2 (\prod_{k=\alpha(r)}^{\omega(r)} a_k) = 0$  by Lemma 3.1 (1). (d) If the length is one,  $I_r$  is either (1) or (2). In both cases  $i_{\alpha(r)}$  is a bottom, and hence  $\psi_1 \psi_1 *_{\varphi_2} \psi_2(a_{\alpha(r)}) = 0$ . Therefore,  $(\varphi_1 \psi_1 *_{\varphi_2} \varphi_2)_{(\psi_1 \psi_1 *_{\varphi_2} \psi_2)} *_{\varphi_3} \varphi_3(a_1 \dots a_n) = 0$  by definition. A similar argument is applicable to  $\varphi_1 \psi_1 *_{(\varphi_2 \psi_2 *_{\varphi_3} \varphi_3)} (\varphi_2 \psi_2 *_{\varphi_3} \varphi_3)(a_1 \dots a_n)$  and then it turns out to be 0. Therefore,  $(\varphi_1 \psi_1 *_{\varphi_2} \varphi_2)_{(\psi_1 \psi_1 *_{\varphi_2} \psi_2)} *_{\varphi_3} \varphi_3 = \varphi_1 \psi_1 *_{(\varphi_2 \psi_2 *_{\varphi_3} \varphi_3)} (\varphi_2 \psi_2 *_{\varphi_3} \varphi_3)$  on  $\mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}_3$ .

We also need to prove that  $(\psi_1 \psi_1 *_{\varphi_2} \psi_2)_{(\psi_1 \psi_1 *_{\varphi_2} \psi_2)} *_{\varphi_3} \varphi_3 = \psi_1 \psi_1 *_{(\varphi_2 \psi_2 *_{\varphi_3} \varphi_3)} (\psi_2 \psi_2 *_{\varphi_3} \varphi_3)$ ; this follows from (3.1) with replacements  $\varphi_1 \mapsto \psi_3$ ,  $\varphi_2 \mapsto \psi_2$ ,  $\varphi_3 \mapsto \psi_1$ ,  $\psi_1 \mapsto \varphi_3$ ,  $\psi_2 \mapsto \varphi_2$  and  $\psi_3 \mapsto \varphi_3$ .

(2) It suffices to prove the equality

$$(\varphi_1 \theta_1 *_{\psi_2} \varphi_2)_{(\theta_1 \theta_1 *_{\psi_2} \theta_2)} *_{\psi_3} \psi_3(a_1 \dots a_n) = \varphi_1 \theta_1 *_{(\psi_2 \psi_2 *_{\varphi_3} \psi_3)} (\varphi_2 \theta_2 *_{\psi_3} \psi_3)(a_1 \dots a_n)$$

for each word  $a_1 \dots a_n$ ,  $(i_1, \dots, i_n) \in D$  and  $a_k \in \mathcal{A}_{i_k}$ . We put an assumption similar to that used in (1):  $\varphi_{i_1}(a_1) = 0$ ,  $a_k \in \text{Ker } \psi_{i_k}$  for  $k \in P(i_1, \dots, i_n) \setminus \{1\}$  and  $a_k \in \text{Ker } \theta_{i_k}$  for  $k \in B(i_1, \dots, i_n) \setminus \{1\}$ . There are two cases where  $i_1 \neq 3$  and  $i_1 = 3$ , and respectively we use the notation  $(i_1 \dots i_n) = (I_1 3 I_2 3 \dots)$  and  $(3 I_1 3 I_2 \dots)$  as used in the proof of (1). If  $i_1 = 3$ , then  $\varphi_{i_1}(a_1) = 0$  by assumption on  $a_1$ . If  $i_1 \neq 3$ , the equality  $\varphi_1 \theta_1 *_{\psi_2} \varphi_2 (\prod_{k=\alpha(1)}^{\omega(1)} a_k) = 0$  follows from Lemma 3.2. The remaining discussion is the same as (1) and  $(\varphi_1 \theta_1 *_{\psi_2} \varphi_2)_{(\theta_1 \theta_1 *_{\psi_2} \theta_2)} *_{\psi_3} \psi_3(a_1 \dots a_n) = 0$  again by Lemma 3.2. In a similar way we obtain  $\varphi_1 \theta_1 *_{(\psi_2 \psi_2 *_{\varphi_3} \psi_3)} (\varphi_2 \theta_2 *_{\psi_3} \psi_3)(a_1 \dots a_n) = 0$ .  $\square$

We now define o-free products, indented products, o-free independence and indented independence.

**Definition 3.4.** (1) Let  $(\mathcal{A}_i, \varphi_i, \psi_i)$ ,  $i = 1, 2, 3, \dots$  be unital algebraic probability spaces equipped with two states. Then the ordered free (o-free) product  $(\mathcal{A}, \varphi, \psi) = \lambda_i(\mathcal{A}_i, \varphi_i, \psi_i)$  is defined by  $\mathcal{A} = *\mathcal{A}_i$  and  $(\varphi, \psi) = \lambda_i(\varphi_i, \psi_i)$ . This is defined without ambiguity since the product  $\lambda$  is associative.

(2) Let  $(\mathcal{A}, \varphi, \psi)$  be a unital algebraic probability space equipped with two states. Let  $\mathcal{A}_i$  be subalgebras of  $\mathcal{A}$  containing the unit of  $\mathcal{A}$ . Then  $\mathcal{A}_i$  are said to be o-free independent if the following property holds for any  $a_k \in \mathcal{A}_{i_k}$  and  $(i_1, \dots, i_n) \in D$ .

(OF)  $\varphi(a_1 \cdots a_n) = 0$  and  $\psi(a_1 \cdots a_n) = 0$  whenever  $\varphi(a_k) = 0$  holds for  $k \in P(i_1, \dots, i_n)$  and  $\psi(a_k) = 0$  holds for  $k \in B(i_1, \dots, i_n)$ .

(3) Let  $(\mathcal{A}_i, \varphi_i, \psi_i, \theta_i)$ ,  $i = 1, 2, 3, \dots$  be unital algebraic probability spaces equipped with three states. Then the indented product  $(\mathcal{A}, \varphi, \psi, \theta) = \lambda_i(\mathcal{A}_i, \varphi_i, \psi_i, \theta_i)$  is defined by  $\mathcal{A} = *\mathcal{A}_i$  and  $(\varphi, \psi, \theta) = \lambda_i(\varphi_i, \psi_i, \theta_i)$ .

(4) Let  $(\mathcal{A}, \varphi, \psi, \theta)$  be a unital algebraic probability space equipped with three states. Let  $\mathcal{A}_i$  be subalgebras of  $\mathcal{A}$  containing the unit of  $\mathcal{A}$ . Then  $\mathcal{A}_i$  are said to be indented independent if the following properties hold for any  $a_k \in \mathcal{A}_{i_k}$  and  $(i_1, \dots, i_n) \in D$ .

(I1)  $\mathcal{A}_i$  are o-free independent w.r.t.  $(\psi, \theta)$ .

(I2)  $\varphi(a_1 \cdots a_n) = 0$  whenever  $\varphi(a_1) = 0$ ,  $\psi(a_k) = 0$  for  $k \in P(i_1, \dots, i_n) \setminus \{1\}$  and  $\theta(a_k) = 0$  for  $k \in B(i_1, i_2, \dots, i_n) \setminus \{1\}$ .

**Remark 3.5.** The reader may wonder why the conditions on  $a_k$  are only put for peaks and bottoms respectively. These conditions are however sufficient to determine all the mixed moments for  $a_k \in \mathcal{A}_{i_k}$  with  $i_1 \neq \dots \neq i_n$  for o-free or indented independent subalgebras  $\mathcal{A}_i$ . This will be cleared in Proposition 3.6.

The indented product generalizes many associative products known in the literature:

$$(\varphi_1, \varphi_1, \varphi_1) \lambda (\varphi_2, \varphi_2, \varphi_2) = (\varphi_1 * \varphi_2, \varphi_1 * \varphi_2, \varphi_1 * \varphi_2) \quad (\text{free product}), \quad (3.2)$$

$$(\varphi_1, \psi_1, \psi_1) \lambda (\varphi_2, \psi_2, \psi_2) = (\varphi_1 \psi_1 * \psi_2, \psi_1 * \psi_2, \psi_1 * \psi_2) \quad (\text{c-free product}), \quad (3.3)$$

$$(\varphi_1, \delta_1, \delta_1) \lambda (\varphi_2, \delta_2, \delta_2) = (\varphi_1 \diamond \varphi_2, \delta_1 * \delta_2, \delta_1 * \delta_2) \quad (\text{Boolean product}), \quad (3.4)$$

$$(\varphi_1, \varphi_1, \delta_1) \lambda (\varphi_2, \varphi_2, \delta_2) = (\varphi_1 \triangleright \varphi_2, \varphi_1 \triangleright \varphi_2, \delta_1 * \delta_2) \quad (\text{monotone product}), \quad (3.5)$$

$$(\varphi_1, \delta_1, \varphi_1) \lambda (\varphi_2, \delta_2, \varphi_2) = (\varphi_1 \triangleleft \varphi_2, \delta_1 * \delta_2, \varphi_1 \triangleleft \varphi_2) \quad (\text{anti-monotone product}), \quad (3.6)$$

$$(\varphi_1, \psi_1, \delta_1) \lambda (\varphi_2, \psi_2, \delta_2) = (\varphi_1 \triangleright_{\psi_2} \varphi_2, \psi_1 \triangleright_{\psi_2} \psi_2, \delta_1 * \delta_2) \quad (\text{c-monotone product}), \quad (3.7)$$

$$(\varphi_1, \delta_1, \psi_1) \lambda (\varphi_2, \delta_2, \psi_2) = (\varphi_1 \psi_1 \triangleleft \varphi_2, \delta_1 * \delta_2, \psi_1 \triangleleft \psi_2) \quad (\text{c-anti-monotone product}). \quad (3.8)$$

An important point here is that the associative laws of the above seven products follow from that of the indented product. Later we show that indented cumulants also generalize the seven kinds of cumulants.

We note that the o-free product generalizes free, monotone and anti-monotone products.

We can put arbitrary conditions on the expectation of  $a_i$ 's for  $i \notin P(i_1, \dots, i_n) \cup B(i_1, \dots, i_n)$  in the definitions of indented independence and o-free independence. The following fact can also be used to characterize the indented and o-free products.

**Proposition 3.6.** Let  $(\mathcal{A}, \varphi, \psi, \theta)$  be a unital algebraic probability space equipped with three states. Let  $\mathcal{A}_i$  be subalgebras of  $\mathcal{A}$  containing the unit of  $\mathcal{A}$ .

(1) Let  $E$  and  $F$  be disjoint subsets of  $\{1, \dots, n\}$  such that  $E \cup F = \{1, \dots, n\}$ ,  $P(i_1, \dots, i_n) \subset E$  and  $B(i_1, \dots, i_n) \subset F$ .  $E$  and  $F$  may depend on  $(i_1, \dots, i_n)$ . Then (I1) is equivalent to the following condition: (I1')  $\psi(a_1 \cdots a_n) = 0$  and  $\theta(a_1 \cdots a_n) = 0$  hold whenever  $a_k \in \mathcal{A}_{i_k}$ ,  $\psi(a_k) = 0$  for any  $k \in E$ ,  $(i_1, \dots, i_n) \in D$  and  $\theta(a_k) = 0$  for any  $k \in F$ .

(2) Let  $E$  and  $F$  be disjoint subsets of  $\{2, \dots, n\}$  such that  $E \cup F = \{2, \dots, n\}$ ,  $P(i_1, \dots, i_n) \setminus \{1\} \subset E$  and  $B(i_1, \dots, i_n) \setminus \{1\} \subset F$ .  $E$  and  $F$  may depend on  $(i_1, \dots, i_n)$ . Then the condition (I2) is equivalent to the following condition: (I2')  $\varphi(a_1 \cdots a_n) = 0$  holds whenever  $a_k \in \mathcal{A}_{i_k}$ ,  $(i_1, \dots, i_n) \in D$ ,  $\varphi(a_1) = 0$ ,  $\psi(a_k) = 0$  for any  $k \in E$  and  $\theta(a_k) = 0$  for  $k \in F$ .

Roughly, we can put any conditions on the kernels at points other than peaks and bottoms. Moreover, we have the following. Let  $E$  and  $F$  be disjoint subsets of  $\{1, \dots, n-1\}$  such that  $E \cup F = \{1, \dots, n-1\}$ ,  $P(i_1, \dots, i_n) \setminus \{n\} \subset E$  and  $B(i_1, \dots, i_n) \setminus \{n\} \subset F$ . Then the condition (I2) is equivalent to the following condition: (I2'')  $\varphi(a_1 \cdots a_n) = 0$  holds whenever  $a_k \in \mathcal{A}_{i_k}$ ,  $\varphi(a_n) = 0$ ,  $(i_1, \dots, i_n) \in D$ ,  $\psi(a_k) = 0$  for  $k \in E$  and  $\theta(a_k) = 0$  for  $k \in F$ .

*Proof.* We notice that (1) follows from (2) under the further assumption  $\varphi = \psi$ . We only prove the equivalence between (I2) and (I2') since the equivalence between (I2) and (I2'') is proved in a similar way.

It is sufficient to prove the implication  $(I2) \Rightarrow (I2')$ ; the converse statement is immediate by definition. If  $l$  is a peak and  $m$  is a bottom such that  $l+1 < m$  and there are no peaks and bottoms in  $\{m+1, \dots, l-1\}$ , then  $i_k$  is an increasing function of  $k$  on  $\{m, \dots, l\}$ . Then  $a_1 \cdots a_n$  can be written as  $a_1 \cdots a_m(a_{m+1} - \lambda_{m+1} + \lambda_{m+1})(a_{m+2} - \lambda_{m+2} + \lambda_{m+2}) \cdots (a_{l-2} - \lambda_{l-1} + \lambda_{l-1})a_l \cdots a_n$  and therefore it can be written by sums and products of  $a_1 a_2 \cdots a_m, (a_{m+1} - \lambda_{m+1}), (a_{m+2} - \lambda_{m+2}), \dots, (a_{l-2} - \lambda_{l-1}), a_l \cdots a_n$ .  $\lambda_k$  is set to be  $\varphi(a_k)$  if  $k = 1$ ,  $\psi(a_k)$  if  $k \in E$  and  $\theta(a_k)$  if  $k \in F$ . Applying this procedure to every increasing part and decreasing part of the function  $i_k$  and taking expectation, we obtain the conclusion.  $\square$

## 4 Realizations of products of states by means of vector states on the free product of Hilbert spaces

In this section we realize o-free independence by taking the free product of Hilbert spaces equipped with unit vectors. Motivation for this section comes from papers [2, 8, 17, 21, 29]. We start from a review of them.

In this section  $\ast \mathcal{B}_k$  denotes the algebraic free product of unital algebras  $\mathcal{B}_k$  with identification of units. Let  $V$  be a Hilbert space. We denote by  $\mathbb{B}(V)$  the set of bounded operators on  $V$ . If  $W$  is a closed subspace of  $V$ ,  $P_W$  denotes the orthogonal projection to  $W$ .

Let  $\mathcal{A}_k$  be a unital  $C^\ast$ -algebra equipped with three states  $(\varphi_k, \psi_k, \theta_k)$  for each  $k$ . We consider  $\ast$ -representations  $\pi_k, \sigma_k, \rho_k : \mathcal{A}_k \rightarrow \mathbb{B}(H_k)$  satisfying  $\varphi_k(a_k) = \langle \pi_k(a_k)\xi_k, \xi_k \rangle$ ,  $\psi_k(a_k) = \langle \sigma_k(a_k)\xi_k, \xi_k \rangle$  and  $\theta_k(a_k) = \langle \rho_k(a_k)\xi_k, \xi_k \rangle$ . We fix a unit vector  $\xi_i$  of  $H_i$  for each  $i$ . We denote by  $H_i^0$  the closed subspace  $H_i \ominus \mathbb{C}\xi_i$ . Let  $(H^F, \xi)$  be the free product of  $(H_i, \xi_i)$  and  $(H^M, \xi)$  the monotone product defined by

$$H^F = \mathbb{C}\xi \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{i_1 \neq \dots \neq i_n} H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0,$$

$$H^M = \mathbb{C}\xi \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{i_1 > \dots > i_n} H_{i_1}^0 \otimes \cdots \otimes H_{i_n}^0.$$

We define  $H^F(k)$  and  $H^M(k)$  by

$$H^F(k) = \mathbb{C}\xi \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1 > \dots > i_n, \\ i_1 \neq k}} H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0,$$

$$H^M(k) = \mathbb{C}\xi \oplus \bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1 > \dots > i_n, \\ i_1 < k}} H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0.$$

$H^F$  and  $H_k \otimes H^F(k)$  are isomorphic by a map  $V_k^F : H^F \rightarrow H_k \otimes H^F(k)$  for each  $k$  defined by  $V_k^F(\xi) = \xi_k \otimes \xi$ ,  $V_k^F(f) = f \otimes \xi$  for  $f \in H_k^0$  and

$$V_k^F(f_1 \otimes \dots \otimes f_n) = \begin{cases} f_1 \otimes (f_2 \otimes \dots \otimes f_n), & i_1 = k, n \geq 2, \\ \xi_k \otimes (f_1 \otimes \dots \otimes f_n), & i_1 \neq k, \end{cases}$$

where  $f_1 \otimes \dots \otimes f_n \in H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0$  with  $i_1 \neq \dots \neq i_n$ . We understand that  $H^M \subset H^F$  and let  $P$  be the orthogonal projection onto  $H^M$ . Then a partial isometry  $V_k^M : H^M \rightarrow H_k \otimes H^M(k)$  can be defined by  $PV_k^F|_{H^M}$ . For instance, the adjoint operator of  $V_k^F$  is written as

$$(V_k^F)^*(v \otimes x) = \langle v, \xi_k \rangle x + P_{H_k^0}(v) \otimes x,$$

where  $v \in H_k$ ,  $x \in H^F(k) \ominus \mathbb{C}\xi$  and  $P_{H_k^0}$  is the orthogonal projection from  $H_k$  to  $H_k^0$ . When  $x = \xi$ ,  $(V_k^F)^*(v \otimes \xi) = \langle v, \xi_k \rangle \xi + P_{H_k^0}(v)$ .

Let  $\lambda_k^X : \mathbb{B}(H_k) \rightarrow \mathbb{B}(H^X)$  ( $X = F, M$ ) be the operators defined by  $\lambda_k^X(A_k) := (V_k^X)^*(A_k \otimes \text{Id}_{H^X(k)})V_k^X$ .  $\lambda_k^X$  ( $X = F, M$ ) are  $*$ -homomorphisms. The action of  $\lambda_k^F(A_k)$  ( $A_k \in \mathbb{B}(H_k)$ ) on  $f_1 \otimes \dots \otimes f_n \in H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0$  is written as

$$\lambda_k^F(A_k)(f_1 \otimes \dots \otimes f_n) = \begin{cases} \langle A_k f_1, \xi_k \rangle f_2 \otimes \dots \otimes f_n + P_{H_k^0}(A_k f_1) \otimes f_2 \otimes \dots \otimes f_n, & i_1 = k, n \geq 2, \\ \langle A_k \xi_k, \xi_k \rangle f_1 \otimes \dots \otimes f_n + P_{H_k^0}(A_k \xi_k) \otimes f_1 \otimes \dots \otimes f_n, & i_1 \neq k. \end{cases}$$

If  $n = 1$  and  $f \in H_k^0$ , then  $\lambda_k^F(A_k)f = \langle A_k f, \xi_k \rangle \xi + P_{H_k^0}(A_k f)$ .  $\lambda_k^M(A_k)$  is expressed similarly. We note that  $\lambda_k^X(A_k)(H_k) \subset H_k$  and  $\lambda_k^X(A_k)(H^X \ominus H_k) \subset H^X \ominus H_k$ .

Let  $J_k^X$  and  $J_k^{CX}$  ( $X = F, M$ ) be respectively defined by  $J_k^X = \lambda_k^X \circ \pi_k$  and  $J_k^{CX}(a_k) = \lambda_k^X(\pi_k(a_k))P_{H_k} \oplus \lambda_k^X(\sigma_k(a_k))P_{H^X \ominus H_k}$ . Moreover, we denote respectively by  $J^F$  and  $J^{CF}$  the natural extensions to the free product  $*\mathcal{A}_k$ .  $J_k^M$  and  $J_k^{CM}$  are non-unital homomorphisms and therefore we extend them to the non-unital free product  $*_{nu}\mathcal{A}_k$ ; these are denoted by  $J^M$  and  $J^{CM}$ , respectively.

The following properties were proved in [2, 8, 17, 21, 29].

**Theorem 4.1.** (1)  $\langle J^F(a)\xi, \xi \rangle = *\varphi_k(a)$  for  $a \in *\mathcal{A}_k$ .

(2)  $\langle J^M(a)\xi, \xi \rangle = \triangleright\varphi_k(a)$  for  $a \in *_{nu}\mathcal{A}_k$ .

(3) Let  $(\varphi^{CF}, \psi^F) := *(\varphi_k, \psi_k)$ . Then  $\langle J^{CF}(a)\xi, \xi \rangle = \varphi^{CF}(a)$  for  $a \in *\mathcal{A}_k$ .

(4) Let  $(\varphi^{CM}, \psi^M) := \triangleright(\varphi_k, \psi_k)$ . Then  $\langle J^{CM}(a)\xi, \xi \rangle = \varphi^{CM}(a)$  for  $a \in *_{nu}\mathcal{A}_k$ .

**Remark 4.2.** All these operators  $J^F$ ,  $J^M$ ,  $J^{CF}$  and  $J^{CM}$  are  $*$ -homomorphisms, and therefore all the products of states are also states.

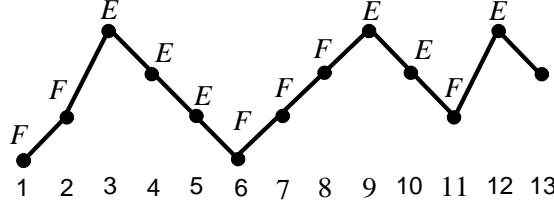


Figure 3:  $i_1 = i_6 = 1$ ,  $i_2 = i_5 = i_7 = i_{11} = 2$ ,  $i_4 = i_8 = i_{10} = i_{13} = 3$ ,  $i_3 = i_9 = i_{12} = 4$ . In this case,  $E = \{3, 4, 5, 9, 10, 12\}$  and  $F = \{1, 2, 6, 7, 8, 11\}$ .

To construct realizations of o-free products and indented products, we introduce four subspaces

$$\begin{aligned}
H_{<}(k) &= \bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1 \neq \dots \neq i_n, \\ i_1 < k}} H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0, \\
H_{=, <}(k) &= \bigoplus_{n=2}^{\infty} \bigoplus_{\substack{i_1 \neq \dots \neq i_n, \\ i_1 = k, i_2 < k}} H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0, \\
H_{>}(k) &= \bigoplus_{n=1}^{\infty} \bigoplus_{\substack{i_1 \neq \dots \neq i_n, \\ i_1 > k}} H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0, \\
H_{=, >}(k) &= \bigoplus_{n=2}^{\infty} \bigoplus_{\substack{i_1 \neq \dots \neq i_n, \\ i_1 = k, i_2 > k}} H_{i_1}^0 \otimes \dots \otimes H_{i_n}^0.
\end{aligned}$$

Then we define invariant subspaces  $H^{OF}(k) = H_{<}(k) \oplus H_{=, <}(k)$  and  $H^{AOF}(k) = H_{>}(k) \oplus H_{=, >}(k)$  of the representation  $\lambda_k$ . We note that  $H^F = H_k \oplus H^{OF}(k) \oplus H^{AOF}(k)$  under the identification of  $\mathbb{C}\xi \oplus H_k^0$  with  $H_k$ . Let  $J_k^I$  be the direct sum of three representations:

$$J_k^I(a_k) = \lambda_k^F(\pi_k(a_k))P_{H_k} \oplus \lambda_k^F(\sigma_k(a_k))P_{H^{OF}(k)} \oplus \lambda_k^F(\rho_k(a_k))P_{H^{AOF}(k)}.$$

$J_k^I$  are all  $*$ -homomorphisms. Denote by  $J^I$  the  $*$ -representation of  $*_k\mathcal{A}_k$  as the natural extension which is defined by using the universal property of the free product as a coproduct.

Let  $J_k^{OF}(a_k)$  be defined by

$$J_k^{OF}(a_k) = \lambda_k^F(\sigma_k(a_k))P_{H_k \oplus H^{OF}(k)} \oplus \lambda_k^F(\theta_k(a_k))P_{H^{AOF}(k)}.$$

This is obtained as a special form of  $J_k^I$  under the further conditions  $\varphi_k = \psi_k$  and  $\pi_k = \sigma_k$ . Let  $J^{OF}$  be the natural extension of  $J_k^{OF}$ 's.

**Theorem 4.3.** (1) Let  $(\psi^{OF}, \theta^{OF})$  be defined by  $\lambda(\psi_k, \theta_k)$ . Then  $\langle J^{OF}(a)\xi, \xi \rangle = \psi^{OF}(a)$  for  $a \in *_k\mathcal{A}_k$ .

(2) Let  $(\varphi^I, \psi^{OF}, \theta^{OF})$  be defined by  $\lambda(\varphi_k, \psi_k, \theta_k)$ . Then  $\langle J^I(a)\xi, \xi \rangle = \varphi^I(a)$  for  $a \in *_k\mathcal{A}_k$ .

*Proof.* We prove the claim for each  $a = a_1 \dots a_n$ ,  $a_k \in \mathcal{A}_{i_k}$ ,  $i_1 \neq \dots \neq i_n$ . It is not difficult to prove the claim for  $n = 1, 2$ . Therefore, we only prove it for  $n \geq 3$ . We take the sets  $E$  and  $F$  in the latter part of Proposition 3.6 (2) for each  $(i_1, \dots, i_n) \in D$  in the following way: (a) if  $k \neq n$  is a peak, then  $k \in E$ ; (b) if  $k \neq n$  is a bottom, then  $k \in F$ ; (c) if  $i_k > i_{k+1}$  and  $1 \leq k \leq n-1$ , we put  $k \in E$ ; (d) if  $i_k < i_{k+1}$  and  $1 \leq k \leq n-1$ , we put  $k \in F$ . This is easily understood in a

diagram and see Fig. 3 for an example. We assume that  $a_m \in \mathcal{A}_{i_m}$  ( $1 \leq m \leq n$ ),  $\varphi_{i_n}(a_n) = 0$ ,  $\psi_{i_k}(a_k) = 0$  for  $k \in E$  and  $\theta_{i_k}(a_k) = 0$  for  $k \in F$ . Then by easy computation, we have

$$J^I(a_1 \cdots a_n)\xi = P_{H_{i_1}^0}(\tau_1(a_1)\xi_{i_1}) \otimes \cdots \otimes P_{H_{i_n}^0}(\tau_n(a_n)\xi_{i_n}),$$

where  $\tau_n = \pi_{i_n}$ ,  $\tau_k = \sigma_{i_k}$  for  $k \in E$  and  $\tau_k = \rho_{i_k}$  for  $k \in F$ . Therefore,  $\langle J^{OF}(a)\xi, \xi \rangle = 0$ . Then the claim follows from Proposition 3.6 (2).  $\square$

## 5 Cumulants

### 5.1 Multivariate cumulants

In this section we define multivariate cumulants, which are sometimes called mixed cumulants or joint cumulants, for o-free independence, and more generally for indented independence. Then we prove the moment-cumulant formulae based on combinatorial structure of linearly ordered non-crossing partitions (Theorem 5.8). The proof shown in this section clarifies many combinatorial structures of linearly ordered non-crossing partitions, but is not so simple. It is expected that the formulae are proved more simply by using the highest coefficients of the products of states. For instance, the moment-cumulant formulae for universal independence (tensor, free, Boolean) can be proved simply on the basis of the highest coefficients (see [13]). If such a method is found for natural products (and for an extension of natural products to the multi-state case), the proof given in this section may be simplified greatly.

Since the two independences are associative it is possible to define cumulants along the line of [13]. A key concept is a *dot operation* which comes from the classical umbral calculus [23].

We outline how to define them without proofs.

**Definition 5.1.** Let  $(\mathcal{A}, \varphi, \psi, \theta)$  be a unital algebraic probability space with three states. We take copies  $\{X^{(j)}\}_{j \geq 1}$  (in an algebraic probability space) for every  $X \in \mathcal{A}$  such that

- (1)  $\varphi(X_1^{(j)} X_2^{(j)} \cdots X_n^{(j)}) = \varphi(X_1 X_2 \cdots X_n)$  for any  $X_i \in \mathcal{A}$ ,  $j, n \geq 1$ ;
- (2) the subalgebras  $\mathcal{A}^{(j)} := \{X^{(j)}\}_{X \in \mathcal{A}}$ ,  $j \geq 1$  are indented independent.

Then we define the dot operation  $N.X$  by

$$N.X = X^{(1)} + \cdots + X^{(N)}$$

for  $X \in \mathcal{A}$  and  $N \in \mathbb{N}$ . We understand that  $0.X = 0$ .

We can iterate the dot operation more than once in a suitable algebraic probability space. Such a space can be constructed in the same idea as in [13].

Similarly we can define the dot operation associated with the o-free product. This is however included in the indented case. In fact, if  $X_1, \dots, X_n$  are indented independent w.r.t.  $(\varphi, \psi, \theta)$ ,  $X_1, \dots, X_n$  are o-free independent w.r.t.  $(\psi, \theta)$ .

**Lemma 5.2.** *The dot operation is associative:*

$$\begin{aligned} \varphi(N.(M.X_1) \cdots N.(M.X_n)) &= \varphi((MN).X_1 \cdots (MN).X_n), \\ \psi(N.(M.X_1) \cdots N.(M.X_n)) &= \psi((MN).X_1 \cdots (MN).X_n), \\ \theta(N.(M.X_1) \cdots N.(M.X_n)) &= \theta((MN).X_1 \cdots (MN).X_n), \end{aligned}$$

for any  $X_i \in \mathcal{A}$ ,  $n \geq 1$ .

**Lemma 5.3.**  $\varphi(N.X_1 \cdots N.X_n)$  is a polynomial of  $N$ ,  $\varphi(X_{i_1} \cdots X_{i_k})$ ,  $\psi(X_{i_1} \cdots X_{i_k})$  and  $\theta(X_{i_1} \cdots X_{i_k})$  ( $i_1 < \cdots < i_k$ ,  $1 \leq k \leq n$ ). This polynomial has no constant terms w.r.t.  $N$ .

By setting a restriction  $\varphi = \psi$  (resp.  $\varphi = \theta$ ), we obtain a similar result for  $\psi(N.X_1 \cdots N.X_n)$  (resp.  $\theta(N.X_1 \cdots N.X_n)$ ).

(resp.  $\psi(X_{i_1} \cdots X_{i_k})$  and  $\theta(X_{i_1} \cdots X_{i_k})$ ) This lemma enables us to define  $\varphi(t.X, \cdots, t.X_n)$  by replacing  $N \in \mathbb{N}$  by  $t \in \mathbb{R}$ .

**Definition 5.4.** Let  $(\mathcal{A}, \varphi, \psi, \theta)$  be an algebraic probability space with three states.

(1) The  $n$ -th o-free cumulant  $K_n^{OF(\psi, \theta)}(X_1, \cdots, X_n)$  (resp. anti-o-free cumulant  $K_n^{AOF(\psi, \theta)}(X_1, \cdots, X_n)$ ) is defined to be the coefficient of  $N$  in  $\psi(N.X_1 \cdots N.X_n)$  (resp.  $\theta(N.X_1 \cdots N.X_n)$ ).

(2) The  $n$ -th indented cumulant  $K_n^{I(\varphi, \psi, \theta)}(X_1, \cdots, X_n)$  is defined by the coefficient of  $N$  in  $\varphi(N.X_1 \cdots N.X_n)$ .

The following properties hold.

(1) (Multilinearity)  $K_n^{I(\varphi, \psi, \theta)}$ ,  $K_n^{OF(\psi, \theta)}$ ,  $K_n^{AOF(\psi, \theta)}$ :  $\mathcal{A}^n \rightarrow \mathbb{C}$  are multilinear.

(2) (Polynomiality) There exist polynomials  $P_n^I$ ,  $P_n^{OF}$  and  $P_n^{AOF}$  such that

$$\begin{aligned} K_n^{I(\varphi, \psi, \theta)}(X_1, \cdots, X_n) &= \varphi(X_1 \cdots X_n) \\ &\quad + P_n^I(\{\varphi(X_{i_1} \cdots X_{i_p}), \psi(X_{i_1} \cdots X_{i_p}), \theta(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \leq p \leq n-1, \\ i_1 < \cdots < i_p}}), \\ K_n^{OF(\psi, \theta)}(X_1, \cdots, X_n) &= \psi(X_1 \cdots X_n) + P_n^{OF}(\{\psi(X_{i_1} \cdots X_{i_p}), \theta(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \leq p \leq n-1, \\ i_1 < \cdots < i_p}}), \\ K_n^{AOF(\psi, \theta)}(X_1, \cdots, X_n) &= \theta(X_1 \cdots X_n) + P_n^{AOF}(\{\psi(X_{i_1} \cdots X_{i_p}), \theta(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \leq p \leq n-1, \\ i_1 < \cdots < i_p}}). \end{aligned}$$

(3) (Extensivity)

$$\begin{aligned} K_n^{I(\varphi, \psi, \theta)}(N.X_1, \cdots, N.X_n) &= N K_n^{I(\varphi, \psi, \theta)}(X_1, \cdots, X_n), \\ K_n^{OF(\psi, \theta)}(N.X_1, \cdots, N.X_n) &= N K_n^{OF(\psi, \theta)}(X_1, \cdots, X_n), \\ K_n^{AOF(\psi, \theta)}(N.X_1, \cdots, N.X_n) &= N K_n^{AOF(\psi, \theta)}(X_1, \cdots, X_n). \end{aligned}$$

(4)  $K_n^{OF(\psi, \theta)} = K_n^{AOF(\theta, \psi)}$ .

The property (1) is proved by observation on the proof of Lemma 5.3 and the property (3) follows from Lemma 5.2. Moreover, we can prove the uniqueness of cumulants under the conditions (1)-(3). The reader is referred to [13] for details. (4) can be proved immediately since the treatment of  $\psi, \theta$  is symmetric.

We introduce notation about partitions of a set to describe a combinatorics of moments and cumulants. An ordered partition of a set  $E$  is a tuple  $\pi = (V_1, \cdots, V_k)$  of disjoint nonempty subsets  $V_1, \cdots, V_k$  of  $E$  such that  $V_1 \cup \cdots \cup V_k = E$ . Each  $V_i$  is called a block of  $\pi$ . This notation is taken from Section 5 of [16]. Let  $\mathcal{LNC}(E)$  be the set of *ordered non-crossing partitions* of  $E$  defined by

$$\mathcal{LNC}(E) = \{\pi = (V_1, \cdots, V_{|\bar{\pi}|}); \bar{\pi} = \{V_1, \cdots, V_{|\bar{\pi}|}\} \in \mathcal{NC}(E)\}.$$

The notation  $\bar{\pi}$  means a partition without an order structure and  $\pi$  means a partition with an order structure.  $|\bar{\pi}|$ , sometimes simply denoted by  $|\pi|$ , is the number of the blocks contained in  $\bar{\pi}$ . We always use this notation in this section. If  $E = \{1, \cdots, n\}$ , we write  $\mathcal{NC}(n)$  and  $\mathcal{LNC}(n)$  instead of  $\mathcal{NC}(E)$  and  $\mathcal{LNC}(E)$ , respectively.



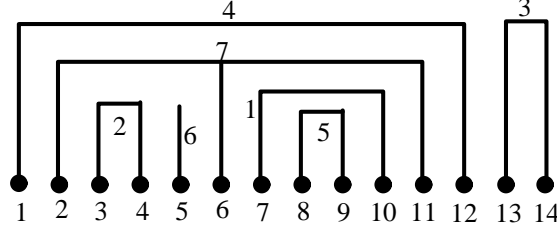


Figure 4: The diagram of  $\pi = (V_1, \dots, V_7) \in \mathcal{LNC}(14)$ ,  $V_1 = \{7, 10\}$ ,  $V_2 = \{3, 4\}$ ,  $V_3 = \{13, 14\}$ ,  $V_4 = \{1, 12\}$ ,  $V_5 = \{8, 9\}$ ,  $V_6 = \{5\}$  and  $V_7 = \{2, 6, 11\}$ .

We denote  $W \succ V$  to express that  $W$  is in the inner side of  $V$ , that is, there exist  $f, g \in V$  such that  $W \subset \{e \in E; f < e < g\}$ . The relation  $\prec$  gives a partial ordering of  $\bar{\pi}$ . We say that a block  $V \in \bar{\pi}$  is *outer* if there is no  $W \in \bar{\pi}$  such that  $V \succ W$ . The set of the outer blocks is denoted as  $\text{Out}(\bar{\pi})$ . A block is called *inner* if it is not outer. The set of the inner blocks is denoted as  $\text{Inn}(\bar{\pi})$ .

We define sets  $S_1(\pi)$ ,  $S_2(\pi)$ ,  $T_1(\pi)$  and  $T_2(\pi)$  for each  $\pi \in \mathcal{LNC}(E)$  in the following way. Let  $V_i$  be a block in  $\bar{\pi}$ .

- (1) If  $V_i$  is outer, then  $V_i \in S_1(\pi)$ .
- (2) Let  $V_j$  be the block in  $\bar{\pi}$  such that  $V_j \prec V_i$  and there is no  $W \in \bar{\pi}$  satisfying  $V_j \prec W \prec V_i$ . If  $j < i$ , then  $V_i \in S_1(\pi)$ . We define  $S_2(\pi) = S_1(\pi)^c$ .
- (3) If  $V_i$  is outer, then  $V_i \in T_2(\pi)$ .
- (4) Let  $V_j$  be the block in  $\bar{\pi}$  such that  $V_j \prec V_i$  and there is no  $W \in \bar{\pi}$  satisfying  $V_j \prec W \prec V_i$ . Moreover if  $j > i$ , then  $V_i \in T_2(\pi)$ .  $T_1(\pi)$  is defined by  $T_2(\pi)^c$ .

An example is shown in Fig. 4. In this example,  $S_1(\pi) = \{V_3, V_4, V_5, V_7\}$ ,  $S_2(\pi) = \{V_1, V_2, V_6\}$ ,  $T_2(\pi) = \{V_1, V_2, V_3, V_4, V_6\}$  and  $T_1(\pi) = \{V_5, V_7\}$ .

We introduce the sets

$$\begin{aligned} \mathcal{LNC}\mathcal{O}(E) &= \{\pi = (V_1, \dots, V_{|\bar{\pi}|}) \in \mathcal{LNC}(E); V_i \succ V_{|\bar{\pi}|} \text{ for all } 1 \leq i \leq |\bar{\pi}| - 1\}, \\ \mathcal{NCO}(E) &= \{\bar{\pi} = \{V_1, \dots, V_{|\bar{\pi}|}\} \in \mathcal{NC}(E); \text{there is a } k \text{ such that } V_i \succ V_k \text{ for all } i \neq k\}. \end{aligned}$$

The former set is called the *linearly ordered non-crossing partitions with the outermost block*, and the latter is called the *non-crossing partitions with the outermost block*.

We prepare notation which is similar to that used in [11].  $\underline{n}$  denotes the set  $\{1, \dots, n\}$ . For a subset  $S \subset \underline{n}$ , let  $\{S_j\}$  be a partition defined as follows. If  $S = \{k_1, \dots, k_m\}$  with  $1 \leq k_1 < \dots < k_m \leq n$ , then  $S_j$  is defined by  $S_j = \{k_{j-1}, \dots, k_j - 1\}$  for  $1 \leq j \leq m + 1$ , where  $k_0 = 1$  and  $k_{m+1} := n$ . If  $k_{j-1} = k_j$ , we understand that  $S_j = \emptyset$ . If  $S = \emptyset$  then  $m = 0$  and  $S_1 = \underline{n}$ . For instance, if  $n = 6$  and  $S = \{1, 2, 4\}$ , then  $S_1 = \emptyset$ ,  $S_2 = \{1\}$ ,  $S_3 = \{2, 3\}$ ,  $S_4 = \{4, 5, 6\}$ . Let  $x_V$  denote the ordered product  $x_{i_1} \cdots x_{i_j}$  for  $V = \{i_1, \dots, i_j\}$ ,  $i_1 < \dots < i_j$ . A product over the empty set is defined to be 1.

The following lemma is similar to that in [11]. This is proved by a simple argument of induction and we omit the proof.

**Lemma 5.5.** *Let  $x_j$  and  $y_k$  be elements of an algebra over  $\mathbb{C}$  with unit 1 and let  $p_j \in \mathbb{C}$ . Then the following identity holds:*

$$x_1 y_1 x_2 y_2 \cdots y_{n-1} x_n = \sum_{S \subset \underline{n}} \left( \prod_{j \notin S} p_j \right) \left( y_{S_1}(x_{k_1} - p_{k_1} 1) \cdots y_{S_m}(x_{k_m} - p_{k_m} 1) y_{S_{m+1}} \right).$$

**Lemma 5.6.** Let  $(\mathcal{A}_i, \varphi_i, \psi_i)$  ( $i = 1, 2$ ) be algebraic probability spaces with two states; let  $(\varphi, \psi)$  be the  $c$ -free product of  $(\varphi_i, \psi_i)$ ; let  $n \geq 2$ . By definition  $\varphi(a_1 b_1 a_2 b_2 \cdots b_{n-1} a_n)$  for  $a_i \in \mathcal{A}_1$  and  $b_i \in \mathcal{A}_2$  can be expressed by sums and products of  $\varphi_1(a_S)$ ,  $\varphi_2(b_U)$ ,  $\psi_1(a_V)$  and  $\psi_2(b_W)$  with  $S, V \subset \underline{n}$  and  $U, W \subset \underline{n-1}$ . Then the term which includes  $\psi_2(b_1 \cdots b_{n-1})$  is given by

$$(\varphi_1(a_1 a_n) - \varphi_1(a_1) \varphi_1(a_n)) \psi_1(a_2) \cdots \psi_1(a_{n-1}) \psi_2(b_1 \cdots b_{n-1}).$$

Moreover, the term which includes  $\varphi_2(b_1 \cdots b_{n-1})$  is given by

$$\varphi_1(a_1) \varphi_1(a_n) \psi_1(a_2) \cdots \psi_1(a_{n-1}) \varphi_2(b_1 \cdots b_{n-1}).$$

*Proof.* Since we only consider coefficients of  $\varphi_2(b_1 \cdots b_{n-1})$  and  $\psi_2(b_1 \cdots b_{n-1})$ , we can assume that  $\varphi_2(b_{i_1} \cdots b_{i_k}) = \psi_2(b_{i_1} \cdots b_{i_k}) = 0$  for all  $i_1 < \cdots < i_k$ ,  $1 \leq k \leq n-2$ . We follow the notation which has appeared in this section. It holds that

$$\begin{aligned} \varphi(a_1 b_1 a_2 b_2 \cdots b_{n-1} a_n) &= \sum_{S \subset \underline{n}} \left( \prod_{j \notin S} \psi_1(a_j) \right) \varphi \left( b_{S_1} (a_{k_1} - \psi_1(a_{k_1}) 1) \cdots (a_{k_m} - \psi_1(a_{k_m}) 1) b_{S_{m+1}} \right) \\ &= \sum_{S=\emptyset, T_1, T_2, T_3} \left( \prod_{j \notin S} \psi_1(a_j) \right) \left( \prod_{j=1}^m (\varphi_1(a_{k_j}) - \psi_1(a_{k_j})) \right) \left( \prod_{j=1}^{m+1} \varphi_2(b_{S_j}) \right), \end{aligned} \quad (5.1)$$

where  $T_1 = \{1\}$ ,  $T_2 = \{n\}$  and  $T_3 = \{1, n\}$ . This is because  $1 \leq |S_j| \leq n-2$  for some  $j$  if  $S \neq \emptyset, T_1, T_2, T_3$ , and therefore, the sum over  $S$  except for  $\emptyset, T_1, T_2, T_3$  becomes 0. The sum over  $\emptyset, T_1, T_2, T_3$  is given by

$$\varphi(a_1 b_1 b_2 \cdots b_{n-1} a_n) \psi_1(a_2) \cdots \psi_1(a_{n-1}),$$

which is equal to

$$\left( (\varphi_1(a_1 a_n) - \varphi_1(a_1) \varphi_1(a_n)) \psi_2(b_1 \cdots b_{n-1}) + \varphi_1(a_1) \varphi_1(a_n) \varphi_2(b_1 \cdots b_{n-1}) \right) \psi_1(a_2) \cdots \psi_1(a_{n-1}).$$

□

Now we derive differential equations which relate moments and cumulants. We fix a linearly ordered finite set  $E = \{e_1, \dots, e_n\}$ ,  $e_1 < \cdots < e_n$  for a while. We define a set  $IB(E)$  consisting of  $V \subset E$  of the form  $V = \{e_k, e_{k+1}, \dots, e_{k+m}\}$ ,  $1 \leq k \leq n$ ,  $0 \leq m \leq n-k$ . For a block  $V = \{e_k, \dots, e_{k+m}\} \in IB(E)$ , we divide  $V^c$  into  $V^c(1) = \{e_1, \dots, e_{k-1}\}$  and  $V^c(2) = \{e_{k+m+1}, \dots, e_n\}$ . If  $k = 1$  (resp.  $k+m = n$ ) we define  $V^c(1) = \emptyset$  (resp.  $V^c(2) = \emptyset$ ).

Let  $\mathcal{I}(E)$  be the set of interval partitions. We embed  $\mathcal{I}(E)$  into  $\mathcal{LNCO}(E)$  and define  $\mathcal{I}(E)$  consisting of partitions  $\pi = (V_1, \dots, V_k)$  satisfying  $V_i \in IB(E)$ ,  $V_1 < \cdots < V_k$ .  $V < W$  means that  $v < w$  for all  $v \in V$  and  $w \in W$ . Moreover, let  $\mathcal{OI}(E)$  be the set of all interval partitions  $\pi$  with  $|\pi|$  odd.  $\mathcal{OI}(E)$  is injectively mapped to  $\mathcal{LNCO}(E)$  by  $(V_1, \dots, V_{2k+1}) \mapsto (V_2, V_4, \dots, V_{2k}, \cup_{p=1}^{k+1} V_{2p-1})$  (see Fig. 5). We denote this image by  $\mathcal{NCIO}(E)$ , an element in which is called a *non-crossing interval partition with the outmost block*. Every partition in  $\mathcal{NCIO}(E)$  arises as follows. Let  $V = \{e_{i_1}, \dots, e_{i_k}\}$  be a subset of  $E$  satisfying  $i_1 = 1$ ,  $i_k = n$  and  $k \geq 2$ . We choose all  $j$  such that  $i_{j+1} > i_j + 1$  and label them  $j(1), \dots, j(r)$ ,  $j(1) < \cdots < j(r)$ . Then  $V_p \in IB(E)$  for  $1 \leq p \leq r$  is defined by  $V_p = \{e_{i_{j(p)}+1}, \dots, e_{i_{j(p)+1}-1}\}$ . We denote  $V$  by  $V_{|\pi|}$  and define  $\pi$  ( $r+1 = |\pi|$ ) by  $\pi = (V_1, \dots, V_{|\pi|-1}, V_{|\pi|})$  which belongs to  $\mathcal{NCIO}(E)$ . The right partition in Fig. 5 is an example.

For simplicity we define a multilinear functional  $\varphi_t : \bigcup_{n \geq 0} \mathcal{A}^n \rightarrow \mathbb{C}$  by

$$\varphi_t(X_1, \dots, X_n) = \varphi(t.X_1 \cdots t.X_n).$$

Similarly we define  $\psi_t$  and  $\theta_t$ . Sometimes it is convenient to write  $\varphi_t(X_V)$  and  $K_k(X_V)$  respectively instead of  $\varphi_t(X_{i_1}, \dots, X_{i_k})$  and  $K_k(X_{i_1}, \dots, X_{i_k})$  for  $V = \{i_1, \dots, i_k\}$ ,  $i_1 < \cdots < i_k$ .

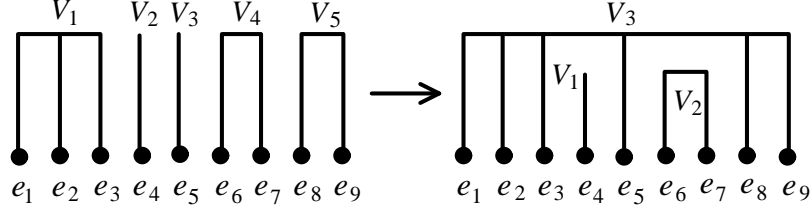


Figure 5: The left figure is  $\pi = (V_1, V_2, V_3, V_4, V_5) \in \mathcal{I}(E)$  and the right is its image of the embedding into  $\mathcal{LNC}(\mathcal{O}(E))$ .

**Proposition 5.7.** *The recurrent differential equations for  $\varphi_t$ ,  $\psi_t$  and  $\theta_t$  are given by*

$$\begin{aligned}
& \frac{d}{dt} \varphi_t(X_1, \dots, X_n) \\
&= \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{NCIO}(V)} \left( \varphi_t(X_{V^c(1)}, X_{V^c(2)}) - \varphi_t(X_{V^c(1)}) \varphi_t(X_{V^c(2)}) \right) \\
&\quad \cdot \theta_t(X_{V_1}) \cdots \theta_t(X_{V_{|\pi|-1}}) K^{OF(\psi, \theta)}(X_{V_{|\pi|}}) \\
&+ \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{NCIO}(V)} \varphi_t(X_{V^c(1)}) \varphi_t(X_{V^c(2)}) \theta_t(X_{V_1}) \cdots \theta_t(X_{V_{|\pi|-1}}) K^{I(\varphi, \psi, \theta)}(X_{V_{|\pi|}}), \\
& \frac{d}{dt} \psi_t(X_1, \dots, X_n) \\
&= \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{NCIO}(V)} \psi_t(X_{V^c(1)}, X_{V^c(2)}) \theta_t(X_{V_1}) \cdots \theta_t(X_{V_{|\pi|-1}}) K^{OF(\psi, \theta)}(X_{V_{|\pi|}}), \\
& \frac{d}{dt} \theta_t(X_1, \dots, X_n) \\
&= \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{NCIO}(V)} \theta_t(X_{V^c(1)}, X_{V^c(2)}) \psi_t(X_{V_1}) \cdots \psi_t(X_{V_{|\pi|-1}}) K^{AOF(\psi, \theta)}(X_{V_{|\pi|}}).
\end{aligned}$$

*Proof.* We recall that  $V \in IB(n)$  divides the set  $\underline{n}$  into three parts  $V^c(1)$ ,  $V$ ,  $V^c(2)$ . For random variables  $X_i$  and  $Y_i$ , we have the identity

$$\begin{aligned}
\varphi((X_1 + Y_1) \cdots (X_n + Y_n)) &= \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{OI}(V)} \varphi(X_{V^c(1)} Y_{V_1} X_{V_2} Y_{V_3} \cdots Y_{V_{|\pi|}} X_{V^c(2)}) \\
&\quad + \varphi(X_1 \cdots X_n).
\end{aligned}$$

With  $X_i$  replaced by  $N.X_i$  and  $Y_i$  by  $(N + M).X_i - N.X_i$ , the above equality becomes

$$\begin{aligned}
& \varphi_{N+M}(X_1, \dots, X_n) \\
&= \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{OI}(V)} \varphi \left( (N.X_{V^c(1)}) ((N + M).X - N.X)_{V_1} N.X_{V_2} ((N + M).X - N.X)_{V_3} \cdots \right. \\
&\quad \left. ((N + M).X - N.X)_{V_{|\pi|}} N.X_{V^c(2)} \right) + \varphi_N(X_1, \dots, X_n),
\end{aligned}$$

where  $N.X_S \equiv (N.X)_S$  denotes  $N.X_{s_1} \cdots N.X_{s_k}$  for  $S = \{s_1, \dots, s_k\}$ ,  $s_1 < \dots < s_k$ . We recall here that  $\{N.X_i\}_i$  and  $\{(N + M).X_i - N.X_i\}_i$  are indented independent and that  $((N + M).X - N.X)_S$  is identically distributed to  $M.X_S$  for any subset  $S \subset \underline{n}$ . The problem of obtaining the

coefficients of  $M$  of every summand then reduces to Lemma 5.6 and it holds that

$$\begin{aligned}
& \varphi_{N+M}(X_1, \dots, X_n) \\
&= \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{NCIO}(V)} \left( \varphi_N(X_{V^c(1)}, X_{V^c(2)}) - \varphi_N(X_{V^c(1)})\varphi_N(X_{V^c(2)}) \right) \\
&\quad \cdot \theta_N(X_{V_1}) \cdots \theta_N(X_{V_{|\pi|-1}}) \psi_M(X_{V_{|\pi|}}) + \varphi_N(X_1, \dots, X_n) \\
&+ \sum_{V \in IB(n)} \sum_{\pi=(V_1, \dots, V_{|\pi|}) \in \mathcal{NCIO}(V)} \varphi_N(X_{V^c(1)})\varphi_N(X_{V^c(2)})\theta_N(X_{V_1}) \cdots \theta_N(X_{V_{|\pi|-1}})\varphi_M(X_{V_{|\pi|}}) + O(M^2) \\
&= M \sum_{V \in IB(n)} \sum_{\pi \in \mathcal{NCIO}(V)} \left( \varphi_N(X_{V^c(1)}, X_{V^c(2)}) - \varphi_N(X_{V^c(1)})\varphi_N(X_{V^c(2)}) \right) \\
&\quad \cdot \theta_N(X_{V_1}) \cdots \theta_N(X_{V_{|\pi|-1}}) K^{OF(\psi, \theta)}(X_{V_{|\pi|}}) + \varphi_N(X_1, \dots, X_n) \\
&+ M \sum_{V \in IB(n)} \sum_{\pi \in \mathcal{NCIO}(V)} \varphi_N(X_{V^c(1)})\varphi_N(X_{V^c(2)})\theta_N(X_{V_1}) \cdots \theta_N(X_{V_{|\pi|-1}}) K^{I(\varphi, \psi, \theta)}(X_{V_{|\pi|}}) + O(M^2).
\end{aligned}$$

We replace  $N, M$  by  $t, s$  respectively and take derivative  $\frac{d}{ds}|_{s=0}$ , and then the first equality follows. The second one follows from the first in the special case  $\varphi = \psi$ , and the third one from the second with the exchange of  $\theta$  and  $\psi$ . We notice that  $K_n^{OF(\psi, \theta)} = K_n^{AOF(\theta, \psi)}$ .  $\square$

The idea of the proof of the following theorem comes from a simple proof of central limit theorem for monotone independence [24].

**Theorem 5.8.** (1) Let  $(\mathcal{A}, \psi, \theta)$  be a unital algebraic probability space endowed with two states. The moment-cumulant formula for  $o$ -free independence is given by

$$\psi(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{LNC}(n)} \frac{1}{|\bar{\pi}|!} \left( \prod_{V \in S_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in S_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right), \quad (5.2)$$

$$\theta(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{LNC}(n)} \frac{1}{|\bar{\pi}|!} \left( \prod_{V \in T_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in T_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right). \quad (5.3)$$

(2) Let  $(\mathcal{A}, \varphi, \psi, \theta)$  be a unital algebraic probability space with three states. The moment-cumulant formula for indented independence is given by (5.2), (5.3) and

$$\begin{aligned}
& \varphi(X_1 \cdots X_n) \\
&= \sum_{\pi \in \mathcal{LNC}(n)} \frac{1}{|\bar{\pi}|!} \left( \prod_{V \in \text{Out}(\bar{\pi})} K_{|V|}^{I(\varphi, \psi, \theta)}(X_V) \right) \left( \prod_{V \in \text{Inn}(\bar{\pi}) \cap S_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in S_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right).
\end{aligned} \quad (5.4)$$

*Proof.* (1) We assume that the formulae

$$\frac{d}{dt} \psi_t(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{LNC}(n)} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}|-1)!} \left( \prod_{V \in S_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in S_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right), \quad (5.5)$$

$$\frac{d}{dt} \theta_t(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{LNC}(n)} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}|-1)!} \left( \prod_{V \in T_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in T_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right) \quad (5.6)$$

hold for  $n \leq N-1$ . We shall prove the formulae for  $n = N$ . Let  $\pi = (V_1, \dots, V_{|\bar{\pi}|}) \in \mathcal{LNC}(N)$ ,  $a = \min\{1 \leq i \leq N; i \in V_{|\bar{\pi}|}\}$  and  $b = \max\{1 \leq i \leq N; i \in V_{|\bar{\pi}|}\}$ . We gather all  $V_j \succ V_{|\bar{\pi}|}$ ,  $V_j \in \bar{\pi}$

(there may be no such  $j$ ); they are denoted by  $\{V_{j_1}, \dots, V_{j_p}\}$ ,  $j_1 < \dots < j_p$ ,  $p \geq 0$ . Then we define an ordered partition  $\sigma = \sigma(\pi) \in \mathcal{LNC}\mathcal{O}(I)$  by  $\sigma = (V_{j_1}, \dots, V_{j_p}, V_{|\bar{\pi}|})$ , where  $I = \{a, a+1, \dots, b\}$ . For instance,  $\sigma = (V_1, V_2, V_5, V_6, V_7)$  in Fig. 4. We gather the blocks  $V_i \notin \bar{\sigma}$ , order them as they appear in  $\pi$  and define  $\sigma^c \in \mathcal{LNC}(I^c)$ . If  $I = \{1, \dots, N\}$ , then we put  $\sigma^c = \emptyset$ . If we neglect the order structure in the above construction, the map

$$\mathcal{NC}(N) \rightarrow \bigcup_{I \subset \{1, \dots, N\}, I \neq \emptyset} (\mathcal{NCO}(I) \times \mathcal{NC}(I^c)), \quad \bar{\pi} \mapsto (\bar{\sigma}, \bar{\sigma}^c)$$

is a bijection. In the existence of the order structure, the map

$$L : \mathcal{LNC}(N) \rightarrow \bigcup_{I \subset \{1, \dots, N\}, I \neq \emptyset} (\mathcal{LNC}\mathcal{O}(I) \times \mathcal{LNC}(I^c)), \quad L(\pi) = (\sigma, \sigma^c) \quad (5.7)$$

is surjective, but not injective. An important point here is that

$$f(\pi) := \left( \prod_{V \in S_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in S_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right)$$

only depends on the image of the map. More precisely, for each  $(\sigma, \sigma^c) \in \mathcal{LNC}\mathcal{O}(I) \times \mathcal{LNC}(I^c)$ , the value  $f(\pi)$ , as a function of  $\pi$ , is constant on  $L^{-1}((\sigma, \sigma^c))$ . In fact,

$$\begin{aligned} f(\pi) &= \left( \prod_{V \in S_1(\sigma) \cup S_1(\sigma^c)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in S_2(\sigma) \cup S_2(\sigma^c)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right) \\ &= f(\sigma)f(\sigma^c). \end{aligned}$$

Moreover,  $|\bar{\pi}|$  is also constant on  $L^{-1}((\sigma, \sigma^c))$ . It is easy to prove that the multiplicity  $|L^{-1}((\sigma, \sigma^c))|$  is equal to  $\frac{(|\bar{\pi}|-1)!}{(|\bar{\sigma}|-1)!|\bar{\sigma}^c|!}$ . Therefore, we can calculate the sum (5.5) for  $n = N$  as

$$\begin{aligned} \sum_{\pi \in \mathcal{LNC}(N)} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}|-1)!} f(\pi) &= \sum_{I \in IB(N)} \left( \sum_{\sigma \in \mathcal{LNC}\mathcal{O}(I)} \frac{t^{|\bar{\sigma}|-1}}{(|\bar{\sigma}|-1)!} f(\sigma) \right) \left( \sum_{\rho \in \mathcal{LNC}(I^c)} \frac{t^{|\bar{\rho}|}}{|\bar{\rho}|!} f(\rho) \right) \\ &= \sum_{I \in IB(N)} \left( \sum_{\sigma \in \mathcal{LNC}\mathcal{O}(I)} \frac{t^{|\bar{\sigma}|-1}}{(|\bar{\sigma}|-1)!} f(\sigma) \right) \psi_t(X_{I^c}). \end{aligned} \quad (5.8)$$

In the second line we used the assumption of induction. Let  $g(\pi)$  be defined by

$$g(\pi) = \left( \prod_{V \in T_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in T_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right).$$

The structure of  $\mathcal{LNC}\mathcal{O}(I)$  is understood by a combination of  $\rho = (W_1, \dots, W_{|\rho|}) \in \mathcal{NCIO}(I)$  and  $\rho_i \in \mathcal{LNC}(W_i)$  for  $1 \leq i \leq |\rho| - 1$  (see Fig. 6); this enables us to calculate

$$\begin{aligned} &\sum_{\sigma = (V_1, \dots, V_{|\bar{\sigma}|}) \in \mathcal{LNC}\mathcal{O}(I)} \frac{t^{|\bar{\sigma}|-1}}{(|\bar{\sigma}|-1)!} f(\sigma) \\ &= \sum_{\rho = (W_1, \dots, W_{|\rho|}) \in \mathcal{NCIO}(I)} K_{|W_{|\rho|}|}^{OF(\psi, \theta)}(X_{W_{|\rho|}}) \sum_{\substack{\rho_r \in \mathcal{LNC}(W_r), \\ 1 \leq r \leq k \equiv |\rho| - 1}} \frac{t^{|\rho_1| + \dots + |\rho_k|}}{|\rho_1|! \dots |\rho_k|!} g(\rho_1) \dots g(\rho_k) \\ &= \sum_{\rho = (W_1, \dots, W_{|\rho|}) \in \mathcal{NCIO}(I)} K_{|W_{|\rho|}|}^{OF(\psi, \theta)}(X_{W_{|\rho|}}) \theta_t(X_{W_1}) \dots \theta_t(X_{W_{|\rho|-1}}). \end{aligned} \quad (5.9)$$

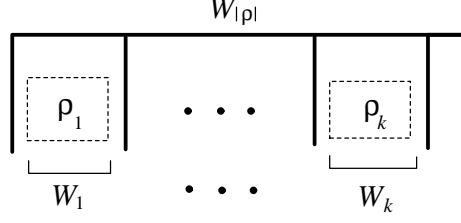


Figure 6: The figure describes  $\sigma = (V_1, \dots, V_{|\sigma|}) \in \mathcal{LNC}\mathcal{O}$  with the use of  $\rho = (W_1, \dots, W_k, W_{|\rho|}) \in \mathcal{NCIO}$  ( $k = |\rho| - 1$ ,  $V_{|\sigma|} = W_{|\rho|}$ ) and  $\rho_i \in \mathcal{NC}(W_i)$ .

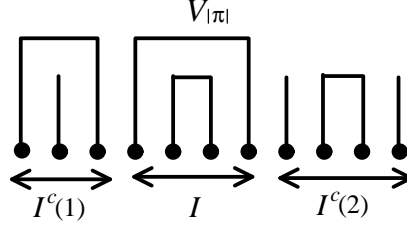


Figure 7: A partition  $\pi = (V_1, \dots, V_{|\pi|})$  with  $V_{|\pi|}$  outer. In this figure  $I = \{4, 5, 6, 7\}$ ,  $I^c(1) = \{1, 2, 3\}$  and  $I^c(2) = \{8, 9, 10, 11\}$ .

Therefore,

$$\sum_{\pi \in \mathcal{LNC}(N)} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}| - 1)!} f(\pi) = \sum_{I \in IB(N)} \sum_{\rho = (V_1, \dots, V_{|\bar{\rho}|}) \in \mathcal{NCIO}(I)} \psi_t(X_{I^c}) K^{OF(\psi, \theta)}(X_{V_{|\bar{\rho}|}}) \theta_t(X_{V_1}) \cdots \theta_t(X_{V_{|\bar{\rho}|-1}}), \quad (5.10)$$

which is then equal to  $\frac{d}{dt} \psi_t(X_1, \dots, X_N)$  by Proposition 5.7. Similarly we can prove the formula for  $\theta_t$ .

(2) A proof similar to (1) is applicable. In addition, we need to divide the sum (5.8) into two parts: if  $V_{|\pi|}$  is an outer block of  $\bar{\pi}$ , the arguments below (5.8) need to be replaced properly. We always use the notation  $\sigma$  to denote the partition constructed in (5.7). We define

$$h(\pi) = \left( \prod_{V \in \text{Out}(\bar{\pi})} K_{|V|}^{I(\varphi, \psi, \theta)}(X_V) \right) \left( \prod_{V \in \text{Inn}(\bar{\pi}) \cap S_1(\pi)} K_{|V|}^{OF(\psi, \theta)}(X_V) \right) \left( \prod_{V \in S_2(\pi)} K_{|V|}^{AOF(\psi, \theta)}(X_V) \right).$$

We assume that

$$\frac{d}{dt} \varphi_t(X_1, \dots, X_n) = \sum_{\pi \in \mathcal{LNC}(n)} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}| - 1)!} h(\pi) \quad (5.11)$$

holds for  $1 \leq n \leq N - 1$ . We notice that

$$\begin{aligned} h(\pi) &= h(\sigma) h(\sigma^c) \text{ if } V_{|\pi|} \in \text{Out}(\bar{\pi}), \\ h(\pi) &= f(\sigma) h(\sigma^c) \text{ if } V_{|\pi|} \in \text{Inn}(\bar{\pi}). \end{aligned}$$

We divide the sum  $\sum_{\pi = (V_1, \dots, V_{|\pi|}) \in \mathcal{LNC}(N)}$  into  $\sum_{\pi \in \mathcal{LNC}(N), V_{|\pi|} \in \text{Out}(\bar{\pi})}$  and  $\sum_{\pi \in \mathcal{LNC}(N), V_{|\pi|} \in \text{Inn}(\bar{\pi})}$ . If  $V_{|\pi|}$  is outer, the partition  $\pi$  is of such a form as shown in Fig. 7. That is, the structure of  $\pi$  with  $V_{|\pi|}$  outer is described as follows. Let  $I \in IB(N)$  and then  $I^c$  has two components  $I^c(1), I^c(2) \in IB(I^c)$  (see the definitions appearing before Proposition 5.7).  $I^c(1)$  or  $I^c(2)$

may be an empty set. Then  $\pi$  consists of three partitions  $(\pi_1, \sigma, \pi_2)$ , where  $\pi_1 \in \mathcal{LNC}(I^c(1))$ ,  $\pi_2 \in \mathcal{LNC}(I^c(2))$ . In addition,  $h(\pi) = h(\pi_1)h(\sigma)h(\pi_2)$ . Therefore, we have

$$\begin{aligned}
& \sum_{\pi \in \mathcal{LNC}(N); V_{|\pi|} \in \text{Out}(\bar{\pi})} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}|-1)!} h(\pi) \\
&= \sum_{I \in IB(N)} \left( \sum_{\pi_1 \in \mathcal{LNC}(I^c(1))} \frac{t^{|\pi_1|}}{|\pi_1|!} h(\pi_1) \right) \left( \sum_{\sigma \in \mathcal{LNC}\mathcal{O}(I)} \frac{t^{|\bar{\sigma}|-1}}{(|\bar{\sigma}|-1)!} h(\sigma) \right) \left( \sum_{\pi_2 \in \mathcal{LNC}(I^c(2))} \frac{t^{|\pi_2|}}{|\pi_2|!} h(\pi_2) \right) \\
&= \sum_{I \in IB(N)} \varphi_t(X_{I^c(1)}) \varphi_t(X_{I^c(2)}) \sum_{\sigma \in \mathcal{LNC}\mathcal{O}(I)} \frac{t^{|\bar{\sigma}|-1}}{(|\bar{\sigma}|-1)!} h(\sigma) \\
&= \sum_{I \in IB(N)} \varphi_t(X_{I^c(1)}) \varphi_t(X_{I^c(2)}) \sum_{\rho=(V_1, \dots, V_{|\bar{\rho}|}) \in \mathcal{NCIO}(I)} K^{I(\varphi, \psi, \theta)}(X_{V_{|\bar{\rho}|}}) \theta_t(X_{V_1}) \cdots \theta_t(X_{V_{|\rho|-1}}).
\end{aligned} \tag{5.12}$$

In the final line we used a relation similar to (5.9).

To calculate the sum over the partitions  $\pi$  with  $V_\pi$  inner, we first calculate the sum over all the partitions and then subtract the sum over  $\pi$  with  $V_\pi$  outer. Let  $\tilde{h}(\pi)$  be defined by

$$\tilde{h}(\pi) = \tilde{h}(\sigma, \sigma^c) = f(\sigma)h(\sigma^c).$$

We note that  $\tilde{h}(\pi) = h(\pi)$  if  $V_{|\pi|} \in \text{Inn}(\bar{\pi})$ . Then

$$\begin{aligned}
& \sum_{\pi \in \mathcal{LNC}(N); V_{|\pi|} \in \text{Inn}(\bar{\pi})} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}|-1)!} h(\pi) \\
&= \sum_{\pi \in \mathcal{LNC}(N)} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}|-1)!} \tilde{h}(\pi) - \sum_{\pi \in \mathcal{LNC}(N); V_{|\pi|} \in \text{Out}(\bar{\pi})} \frac{t^{|\bar{\pi}|-1}}{(|\bar{\pi}|-1)!} h(\pi) \\
&= \sum_{I \in IB(N)} \left( \sum_{\sigma \in \mathcal{LNC}\mathcal{O}(I)} \frac{t^{|\bar{\sigma}|-1}}{(|\bar{\sigma}|-1)!} f(\sigma) \right) \varphi_t(X_{I^c}) \\
&\quad - \sum_{I \in IB(N)} \left( \sum_{\pi_1 \in \mathcal{LNC}(I^c(1))} \frac{t^{|\pi_1|}}{|\pi_1|!} h(\pi_1) \right) \left( \sum_{\sigma \in \mathcal{LNC}\mathcal{O}(I)} \frac{t^{|\bar{\sigma}|-1}}{(|\bar{\sigma}|-1)!} f(\sigma) \right) \left( \sum_{\pi_2 \in \mathcal{LNC}(I^c(2))} \frac{t^{|\pi_2|}}{|\pi_2|!} h(\pi_2) \right) \\
&= \sum_{I \in IB(N)} \sum_{\rho=(V_1, \dots, V_{|\bar{\rho}|}) \in \mathcal{NCIO}(I)} \varphi_t(X_{I^c}) K^{OF(\psi, \theta)}(X_{V_{|\bar{\rho}|}}) \theta_t(X_{V_1}) \cdots \theta_t(X_{V_{|\rho|-1}}) \\
&\quad - \sum_{I \in IB(N)} \varphi_t(X_{I^c(1)}) \varphi_t(X_{I^c(2)}) \sum_{\rho=(V_1, \dots, V_{|\bar{\rho}|}) \in \mathcal{NCIO}(I)} K^{OF(\psi, \theta)}(X_{V_{|\bar{\rho}|}}) \theta_t(X_{V_1}) \cdots \theta_t(X_{V_{|\rho|-1}}).
\end{aligned} \tag{5.13}$$

We used a relation similar to (5.8) in the second equality and relations similar to (5.10) and (5.12) in the last equality. Therefore the equality (5.11) holds for  $n = N$  by Proposition 5.7.  $\square$

In the literature, moment-cumulant formulae were proved for free, c-free, monotone, anti-monotone, c-monotone (only for single variable) and Boolean independences; see [11, 13, 14, 26, 28]. The anti-monotone case is essentially the same as the monotone case. As is expected, the moment-cumulant formula for indented independence generalizes these results. For instance, we explain the c-monotone case which is a somewhat new result. If  $\mathcal{A}$  admits a decomposition  $\mathcal{A} = \mathbb{C}1 \oplus \mathcal{A}^0$  where  $\mathcal{A}^0$  is a  $*$ -algebra, we define  $\theta = \delta$ . In this case  $K_n^{CM(\varphi, \psi)} := K_n^{I(\varphi, \psi, \delta)}$

is the  $n$ -th cumulant for c-monotone independence. Moreover,  $K_n^{M(\psi)} := K_n^{OF(\psi, \delta)}$  is the  $n$ -th monotone cumulant and  $K_n^{AOF(\psi, \delta)} = 0$  on  $\mathcal{A}^0$ . Therefore, only the sum over partitions  $\pi$  satisfying  $S_2(\pi) = \emptyset$  remains. Such a partition is no other than a *monotone partition* [13, 16, 19]; the set of monotone partitions is defined by

$$\mathcal{M}(n) = \{\pi = (V_1, \dots, V_{|\pi|}) \in \mathcal{LNC}(n); \text{ if } V_i \succ V_j, \text{ then } i > j\}$$

as a subset of  $\mathcal{LNC}(n)$ . The moment-cumulant formula for c-monotone independence is obtained from (5.4) as follows.

**Corollary 5.9.** *The moment-cumulant formulae for c-monotone independence and monotone independence are*

$$\begin{aligned} \varphi(X_1 \cdots X_n) &= \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} \left( \prod_{V \in \text{Out}(\bar{\pi})} K_{|V|}^{CM(\varphi, \psi)}(X_V) \right) \left( \prod_{V \in \text{Inn}(\bar{\pi})} K_{|V|}^{M(\psi)}(X_V) \right), \\ \psi(X_1 \cdots X_n) &= \sum_{\pi \in \mathcal{M}(n)} \frac{1}{|\pi|!} \prod_{V \in \bar{\pi}} K_{|V|}^{M(\psi)}(X_V). \end{aligned}$$

Similarly, we can prove the following results obtained in the literature. In the c-free, free and Boolean cases, the summands do not depend on the order structure of  $\mathcal{LNC}(n)$ , and therefore, the factor  $\frac{1}{|\pi|!}$  vanishes after taking the partial sum over the linear order structure.

**Corollary 5.10.** (1)  $K_n^{CF(\varphi, \psi)} := K_n^{I(\varphi, \psi, \psi)}$  is the  $n$ -th c-free cumulant and  $K_n^{F(\psi)} := K_n^{I(\psi, \psi, \psi)}$  is the  $n$ -th free cumulant:

$$\begin{aligned} \varphi(X_1 \cdots X_n) &= \sum_{\pi \in \mathcal{NC}(n)} \left( \prod_{V \in \text{Out}(\pi)} K_{|V|}^{CF(\varphi, \psi)}(X_V) \right) \left( \prod_{V \in \text{Inn}(\pi)} K_{|V|}^{F(\psi)}(X_V) \right), \\ \psi(X_1 \cdots X_n) &= \sum_{\pi \in \mathcal{NC}(n)} \prod_{V \in \pi} K_{|V|}^{F(\psi)}(X_V). \end{aligned}$$

(2)  $K_n^{CAM(\varphi, \psi)} := K_n^{I(\varphi, \delta, \psi)}$  is the  $n$ -th c-anti-monotone cumulant and  $K_n^{AM(\psi)} := K_n^{I(\psi, \delta, \psi)}$  is the  $n$ -th anti-monotone cumulant:

$$\begin{aligned} \varphi(X_1 \cdots X_n) &= \sum_{\pi \in \mathcal{AM}(n)} \frac{1}{|\pi|!} \left( \prod_{V \in \text{Out}(\bar{\pi})} K_{|V|}^{CAM(\varphi, \psi)}(X_V) \right) \left( \prod_{V \in \text{Inn}(\bar{\pi})} K_{|V|}^{AM(\psi)}(X_V) \right), \\ \psi(X_1 \cdots X_n) &= \sum_{\pi \in \mathcal{AM}(n)} \frac{1}{|\pi|!} \prod_{V \in \bar{\pi}} K_{|V|}^{AM(\psi)}(X_V), \end{aligned}$$

where  $\mathcal{AM}(n)$  is the set of the anti-monotone partitions [19] defined by

$$\mathcal{AM}(n) = \{\pi = (V_1, \dots, V_{|\pi|}) \in \mathcal{LNC}(n); \text{ if } V_i \succ V_j, \text{ then } i < j\}.$$

(3)  $K_n^{B(\varphi)} := K_n^{I(\varphi, \delta, \delta)}$  is the  $n$ -th Boolean cumulant:

$$\varphi(X_1 \cdots X_n) = \sum_{\pi \in \mathcal{I}(n)} \prod_{V \in \pi} K_{|V|}^{B(\varphi)}(X_V).$$

**Remark 5.11.** Many probabilistic objects in the literature seem to be isomorphic between anti-monotone independence and monotone independence. This is the case for cumulants: no difference appears between monotone cumulants and anti-monotone cumulants. More precisely, since  $\mathcal{AM}(n)$  is obtained from  $\mathcal{M}(n)$  only by reversing the order, anti-monotone cumulants and monotone cumulants coincide. For the same reason, c-monotone cumulants and c-anti-monotone cumulants also coincide.



## 5.2 Cumulants for single variable

A relation between generating functions of moments and cumulants can be calculated from differential equations in the case of single variable; this method was used in [11]. This is important to calculate limit distributions in central limit theorem. In this section we focus on the cumulants of single variable. When  $X_i$ 's are all set to be a single variable  $X$  in Definition 5.4, we can define the indented cumulants  $K_n^{I(\varphi, \psi, \theta)}(X)$ , the o-free cumulants  $K_n^{OF(\psi, \theta)}(X)$  and the anti-o-free cumulants  $K_n^{AOF(\psi, \theta)}(X)$  of  $X$ . We focus on self-adjoint operators, and hence, we may use the indented and o-free convolutions of probability distributions to describe the sum of identically distributed, independent random variables.

If  $X$  has distributions  $(\lambda, \mu, \nu)$  w.r.t. the states  $(\varphi, \psi, \theta)$ , we denote  $K_n^{I(\varphi, \psi, \theta)}(X)$  by  $K_n^I(\lambda, \mu, \nu)$ ,  $K_n^{OF(\psi, \theta)}(X)$  by  $K_n^{OF}(\mu, \nu)$  and  $K_n^{AOF(\psi, \theta)}(X)$  by  $K_n^{AOF}(\mu, \nu)$ .

Let  $\{(\lambda_t, \mu_t, \nu_t)\}_{t \geq 0}$  be an indented convolution semigroup with  $(\lambda_0, \mu_0, \nu_0) = (\delta_0, \delta_0, \delta_0)$ .  $\lambda_t, \mu_t$  and  $\nu_t$  are assumed to have all finite moments for all  $t > 0$ . We assume that the coefficients of the formal power series of  $F_{\lambda_t}, F_{\mu_t}$  and  $F_{\nu_t}$  are all differentiable w.r.t.  $t$ . we define  $A_\lambda(z) := \frac{\partial F_{\lambda_t}(z)}{\partial t}|_{t=0}$ ,  $B_\mu(z) := \frac{\partial F_{\mu_t}(z)}{\partial t}|_{t=0}$  and  $B_\nu(z) := \frac{\partial F_{\nu_t}(z)}{\partial t}|_{t=0}$ . By Proposition 2.1,

$$F_{\lambda_t \nu_t \boxplus \mu_s \lambda_s} = F_{\lambda_{t+s}} = F_{\lambda_t} \circ F_{\nu_t}^{-1} \circ F_{\nu_t \boxplus \mu_s} + F_{\lambda_s} \circ F_{\mu_s}^{-1} \circ F_{\nu_t \boxplus \mu_s} - F_{\nu_t \boxplus \mu_s}. \quad (5.14)$$

We can then derive differential equations which connect moments and cumulants.

**Proposition 5.12.** *The following differential equations hold.*

$$\frac{\partial F_{\lambda_t}}{\partial t} = A_\lambda \circ F_{\mu_t} - C_\nu \circ F_{\mu_t} + (C_\nu \circ F_{\mu_t}) \cdot \frac{\partial F_{\lambda_t}}{\partial z}, \quad (5.15)$$

$$\frac{\partial F_{\lambda_t}}{\partial t} = A_\lambda \circ F_{\nu_t} - B_\mu \circ F_{\nu_t} + (B_\mu \circ F_{\nu_t}) \cdot \frac{\partial F_{\lambda_t}}{\partial z}, \quad (5.16)$$

$$\frac{\partial F_{\lambda_t}}{\partial t} = \frac{(B_\mu \circ F_{\nu_t}) \cdot (A_\lambda \circ F_{\mu_t}) - (A_\lambda \circ F_{\nu_t}) \cdot (C_\nu \circ F_{\mu_t})}{B_\mu \circ F_{\nu_t} - C_\nu \circ F_{\mu_t}}. \quad (5.17)$$

(5.17) is valid only when the denominator is not zero.

*Proof.* We differentiate the equality (5.14) w.r.t.  $t$ , which yields

$$\begin{aligned} \frac{\partial F_{\lambda_s}}{\partial s} &= A_\lambda \circ F_{\mu_s} + \frac{\partial}{\partial t} \Big|_{t=0} F_{\nu_t}^{-1} \circ F_{\nu_t \boxplus \mu_s} + \frac{\partial F_{\lambda_s}}{\partial z} \cdot \frac{\partial (F_{\mu_s}^{-1})}{\partial z} \circ F_{\mu_s} \cdot \frac{\partial}{\partial t} \Big|_{t=0} F_{\nu_t \boxplus \mu_s} - \frac{\partial}{\partial t} \Big|_{t=0} F_{\nu_t \boxplus \mu_s} \\ &= A_\lambda \circ F_{\mu_s} + \left( \frac{\partial}{\partial t} \Big|_{t=0} F_{\nu_t}^{-1} \right) \circ F_{\mu_s} + \frac{\partial F_{\lambda_s}}{\partial z} \cdot \frac{\partial (F_{\mu_s}^{-1})}{\partial z} \circ F_{\mu_s} \cdot \frac{\partial}{\partial t} \Big|_{t=0} F_{\nu_t \boxplus \mu_s}. \end{aligned}$$

From (2.9), the relation  $F_{\nu_t \boxplus \mu_s}^{-1} = F_{\nu_t}^{-1} + F_{\mu_s}^{-1}$  holds and then this leads to

$$\frac{\partial}{\partial t} \Big|_{t=0} F_{\nu_t \boxplus \mu_s} = (C_\nu \circ F_{\mu_s}) \cdot \frac{\partial F_{\mu_s}}{\partial z}$$

after simple calculations. We note that  $\frac{\partial (F_{\mu_s}^{-1})}{\partial z} \circ F_{\mu_s} = \frac{1}{\frac{\partial F_{\mu_s}}{\partial z}}$  and  $\frac{\partial}{\partial t} \Big|_{t=0} F_{\nu_t}^{-1} = -C_\nu$ . With these, the equality (5.15) holds. (5.16) follows from the replacement of  $(\mu_t, \nu_t)$  with  $(\nu_t, \mu_t)$ . (5.17) comes from  $(B_\mu \circ F_{\nu_t}) \times (5.15) - (C_\nu \circ F_{\mu_t}) \times (5.16)$ .  $\square$

**Corollary 5.13.** *The following differential equations hold.*

$$\frac{\partial F_{\mu_t}}{\partial t} = (B_\mu \circ F_{\nu_t}) \cdot \frac{\partial F_{\mu_t}}{\partial z}, \quad (5.18)$$

$$\frac{\partial F_{\nu_t}}{\partial t} = (C_\nu \circ F_{\mu_t}) \cdot \frac{\partial F_{\nu_t}}{\partial z}, \quad (5.19)$$

$$\frac{\partial F_{\mu_t}}{\partial t} = B_\mu \circ F_{\mu_t} - C_\nu \circ F_{\mu_t} + (C_\nu \circ F_{\mu_t}) \cdot \frac{\partial F_{\mu_t}}{\partial z}, \quad (5.20)$$

$$\frac{\partial F_{\nu_t}}{\partial t} = C_\nu \circ F_{\nu_t} - B_\mu \circ F_{\nu_t} + (B_\mu \circ F_{\nu_t}) \cdot \frac{\partial F_{\nu_t}}{\partial z}, \quad (5.21)$$

$$\frac{\partial F_{\mu_t}}{\partial t} = \frac{(B_\mu \circ F_{\nu_t})(B_\mu \circ F_{\mu_t} - C_\nu \circ F_{\mu_t})}{B_\mu \circ F_{\nu_t} - C_\nu \circ F_{\mu_t}}, \quad (5.22)$$

$$\frac{\partial F_{\nu_t}}{\partial t} = \frac{(C_\nu \circ F_{\mu_t})(B_\mu \circ F_{\nu_t} - C_\nu \circ F_{\nu_t})}{B_\mu \circ F_{\nu_t} - C_\nu \circ F_{\mu_t}}. \quad (5.23)$$

(5.22) and (5.23) are valid only when the denominator is not zero.

*Proof.* We notice that if  $\{(\mu_t, \nu_t)\}_{t \geq 0}$  is an o-free convolution semigroup, both  $\{(\mu_t, \mu_t, \nu_t)\}_{t \geq 0}$  and  $\{(\nu_t, \mu_t, \nu_t)\}_{t \geq 0}$  become indented convolution semigroups. (5.18), (5.19), (5.20), (5.21), (5.22) and (5.23) follow from the restrictions  $\lambda_t = \mu_t$  in (5.16),  $\lambda_t = \nu_t$  in (5.15),  $\lambda_t = \mu_t$  in (5.15),  $\lambda_t = \nu_t$  in (5.16),  $\lambda_t = \mu_t$  in (5.17) and  $\lambda_t = \nu_t$  in (5.17), respectively.  $\square$

The above discussions were done for a convolution semigroup. Now we change the viewpoint: assume that formal power series

$$A_\lambda(z) = - \sum_{n=1}^{\infty} \frac{K_n^I(\lambda, \mu, \nu)}{z^{n-1}}, \quad B_\mu(z) = - \sum_{n=1}^{\infty} \frac{K_n^{OF}(\mu, \nu)}{z^{n-1}}, \quad C_\nu(z) = - \sum_{n=1}^{\infty} \frac{K_n^{AOF}(\mu, \nu)}{z^{n-1}}$$

are given for probability measures  $(\lambda, \mu, \nu)$  with finite moments of all orders. We consider the initial value problem

$$\frac{\partial F_\lambda}{\partial t} = A_\lambda \circ F_\mu - C_\nu \circ F_\mu + C_\nu \circ F_\mu \frac{\partial F_\lambda}{\partial z}, \quad (5.24)$$

$$\frac{\partial F_\mu}{\partial t} = B_\mu \circ F_\nu \frac{\partial F_\mu}{\partial z}, \quad (5.25)$$

$$\frac{\partial F_\nu}{\partial t} = C_\nu \circ F_\mu \frac{\partial F_\nu}{\partial z}, \quad (5.26)$$

with  $F_\lambda(0, z) = z$ ,  $F_\mu(0, z) = z$  and  $F_\nu(0, z) = z$ .

We define  $G_\rho(t, z) = \frac{1}{F_\rho(t, z)}$  for  $\rho = \lambda, \mu, \nu$  and look for solutions as formal power series of the forms  $\sum_{n=0}^{\infty} \frac{M_n^\rho(t)}{z^{n+1}}$ , where  $M_n^\rho(t)$  are polynomials of  $t$ . We can easily prove the existence and uniqueness of solutions. The equations for  $G_\lambda(t, z)$ ,  $G_\mu(t, z)$  and  $G_\nu(t, z)$  turn out to be the special case of the equations in Proposition 5.7. Therefore we immediately obtain  $F_\rho(1, z) = F_\rho(z)$  for  $\rho = \lambda, \mu, \nu$ .

It is important to note that the equations (5.25) and (5.26) can be replaced with

$$\frac{\partial F_\mu}{\partial t} = B_\mu \circ F_\mu - C_\nu \circ F_\mu + (C_\nu \circ F_\mu) \cdot \frac{\partial F_\mu}{\partial z}, \quad (5.27)$$

$$\frac{\partial F_\nu}{\partial t} = C_\nu \circ F_\nu - B_\mu \circ F_\nu + (B_\mu \circ F_\nu) \cdot \frac{\partial F_\nu}{\partial z}, \quad (5.28)$$

since both pairs of equations define the solutions  $F_\mu(t, z)$  and  $F_\nu(t, z)$  uniquely, and both have been derived from the same functional relation (5.14).

We notice that the equations (5.24)-(5.26) can be understood to be relations between moment generating functions and cumulant generating functions.

**Remark 5.14.** (1) The differential equations (5.15) and (5.16) are identical if  $\lambda_t = \mu_t = \nu_t$  for all  $t > 0$ . In this case they are the complex Burgers equation

$$\frac{\partial F_{\mu_t}}{\partial t}(z) = B_\mu(F_{\mu_t}(z)) \frac{\partial F_{\mu_t}}{\partial z}(z)$$

derived in [30]. If  $\lambda_t = \mu_t$  and  $\nu_t = \delta_0$ , the differential equations are not identical and they become

$$\frac{\partial F_{\mu_t}}{\partial t}(z) = B_\mu(F_{\mu_t}(z)), \quad \frac{\partial F_{\mu_t}}{\partial t}(z) = B_\mu(z) \frac{\partial F_{\mu_t}}{\partial z}(z),$$

which appeared in the monotone case [17].

## 6 Central limit theorem

In this section we prove the central limit theorems for o-free and indented independences. Other limit theorems such as Poisson's law of small numbers can be formulated; it is however difficult to calculate the explicit forms of the density functions and we only prove the central limit theorems here.

**Theorem 6.1.** *Let  $(\mathcal{A}, \varphi, \psi, \theta)$  be a unital  $C^*$ -algebraic probability space with three states. Let  $\{X_i\}_{i=1}^\infty$  be identically distributed (w.r.t. each state), o-free independent and self-adjoint random variables in  $\mathcal{A}$ . If  $\varphi(X_i) = \psi(X_i) = \theta(X_i) = 0$ ,  $\varphi(X_i^2) = \alpha^2$ ,  $\psi(X_i^2) = \beta^2$  and  $\theta(X_i^2) = \gamma^2$ , then the distribution of  $\frac{X_1 + \dots + X_n}{\sqrt{n}}$  w.r.t.  $(\varphi, \psi, \theta)$  converges to a triple of Kesten distributions  $(\lambda, \mu, \nu)$  characterized by*

$$F_\lambda(z) = \left(1 - \frac{\alpha^2}{\beta^2 + \gamma^2}\right)z + \frac{\alpha^2}{\beta^2 + \gamma^2} \sqrt{z^2 - 2(\beta^2 + \gamma^2)}, \quad (6.1)$$

$$F_\mu(z) = \left(1 - \frac{\beta^2}{\beta^2 + \gamma^2}\right)z + \frac{\beta^2}{\beta^2 + \gamma^2} \sqrt{z^2 - 2(\beta^2 + \gamma^2)}, \quad (6.2)$$

$$F_\nu(z) = \left(1 - \frac{\gamma^2}{\beta^2 + \gamma^2}\right)z + \frac{\gamma^2}{\beta^2 + \gamma^2} \sqrt{z^2 - 2(\beta^2 + \gamma^2)}. \quad (6.3)$$

In particular,  $(\mu, \nu)$  is the limit distributions of central limit theorem for o-free independence.

**Remark 6.2.** The limit distributions are Kesten distributions; this result generalizes the limit distributions in the c-free and c-monotone cases.

*Proof.* The moments  $\varphi((\frac{X_1 + \dots + X_n}{\sqrt{n}})^k)$ ,  $\psi((\frac{X_1 + \dots + X_n}{\sqrt{n}})^k)$  and  $\theta((\frac{X_1 + \dots + X_n}{\sqrt{n}})^k)$  respectively converge to the moments  $m_k(\lambda)$ ,  $m_k(\mu)$  and  $m_k(\nu)$ , where  $(\lambda, \mu, \nu)$  are characterized by the cumulants  $(K_2^I(\lambda, \mu, \nu), K_2^{OF}(\mu, \nu), K_2^{AOF}(\mu, \nu)) = (\alpha^2, \beta^2, \gamma^2)$  and  $(K_n^I(\lambda, \mu, \nu), K_n^{OF}(\mu, \nu), K_n^{AOF}(\mu, \nu)) = (0, 0, 0)$  for  $n = 1, n \geq 3$ . This fact comes from the additivity and homogeneity of cumulants; see [11] for detailed discussions.

We first show the limit theorem for o-free independence, i.e., for the distribution of  $X_i$  w.r.t.  $(\psi, \theta)$ .

The limit measures can be calculated by solving the differential equations (5.22) and (5.23) with  $B_\mu(z) = -\frac{\beta^2}{z}$  and  $C_\nu(z) = -\frac{\gamma^2}{z}$ . We assume that  $\beta^2 \neq \gamma^2$  and then the denominators

are not zero. These equations give holomorphic solutions outside a ball and therefore the limit moments come from compactly supported measures  $(\mu, \nu)$ . Therefore, the convergence is in fact in the sense of weak convergence.

Now we calculate the limit distributions. (5.22) becomes

$$\frac{\partial F_\mu}{\partial t}(t, z) = \frac{\beta^2(\beta^2 - \gamma^2)}{-\beta^2 F_\mu(t, z) + \gamma^2 F_\nu(t, z)} \quad (6.4)$$

and (5.23) becomes

$$\frac{\partial F_\nu}{\partial t}(t, z) = \frac{\gamma^2(\beta^2 - \gamma^2)}{-\beta^2 F_\mu(t, z) + \gamma^2 F_\nu(t, z)}. \quad (6.5)$$

Therefore, we have  $\gamma^2 \frac{\partial F_\mu}{\partial t}(t, z) = \beta^2 \frac{\partial F_\nu}{\partial t}(t, z)$ , which implies that

$$F_{\mu_t}(z) = s F_{\nu_t}(z) + (1 - s)z, \quad (6.6)$$

where  $s = \frac{\beta^2}{\gamma^2}$ . After simple calculations, we obtain

$$F_{\mu_t}(z) = \left(1 - \frac{s}{1+s}\right)z + \frac{s}{1+s} \sqrt{z^2 - 2\gamma^2(1+s)t}, \quad (6.7)$$

$$F_{\nu_t}(z) = \left(1 - \frac{1}{1+s}\right)z + \frac{1}{1+s} \sqrt{z^2 - 2\gamma^2(1+s)t}. \quad (6.8)$$

The limit distributions are given by  $(\mu, \nu) = (\mu_1, \nu_1)$ .

$\lambda$  is calculated by the relation (5.17) which yields

$$\frac{\partial F_\lambda}{\partial t}(t, z) = \frac{\alpha^2(\beta^2 - \gamma^2)}{-\beta^2 F_\mu(t, z) + \gamma^2 F_\nu(t, z)}.$$

By simple calculation we obtain the conclusion.

The limit distributions do not have a singular point at  $\beta^2 = \gamma^2$ . If  $\beta^2 = \gamma^2$ , we take a sequence  $\beta_n$  such that  $\beta_n^2 \neq \gamma^2$  and  $\beta_n^2$  converges to  $\gamma^2$ . Since all the moments are determined by variance and continuously depend on variance, we have the weak convergence of the distributions and the same formulae (6.1)-(6.3) hold.  $\square$

## Acknowledgements

The author expresses his sincere gratitude to Mr. Hayato Saigo for suggesting independence in three states and for discussions about quantum probability, multivariate cumulants and umbral calculus. He is grateful to Professor Izumi Ojima for indicating improvements of the manuscript and for discussions about independence. He thanks Professor Uwe Franz for informing him a proof of the associativity of the c-free product. He is also grateful to Mr. Hiroshi Ando for discussions about von Neumann algebras and independence. He thanks Professor Naofumi Muraki for discussions about difference of monotone and anti-monotone independence. He thanks Professor Marek Bożejko for indicating many suggestions for future research and informing the author the reference [3]. This work was supported by JSPS KAKENHI 21-5106.

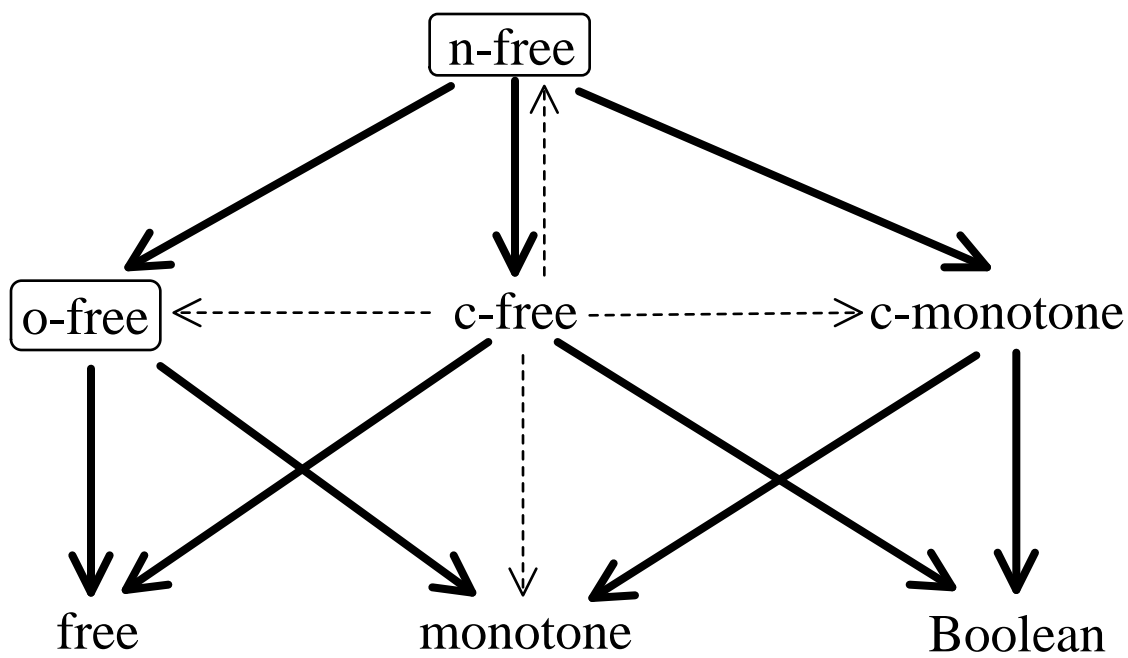
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